WHY MATH EXPRESSIONS HAS ALWAYS BEEN A CCSS PROGRAM

Many aspects of Math Expressions were based on the research and recommendations in National Research Council reports. Other aspects like grade-level placements and representations were drawn from programs in countries that are successful in mathematics; these were then tested in and adapted to U.S. classrooms during the Children's Math Worlds (CMW) Research Project that produced the Math Expressions program. The Common Core State Standards (CCSS) were developed by drawing on many of the same sources. So it is not surprising that the first Math Expressions program met grade-level standards in the CCSS and also supported the research-based learning paths within and across grades that exist in the CCSS.

The CCSS approaches that were so successful in the first Math Expressions program are in the Math Expressions © 2013 Common Core. Lessons that met state standards but not CCSS standards were dropped in making the Math Expressions © 2013.

Common Core. This allowed even more time to be spent on core grade-level standards.

The CCSS are focused and coherent. They carefully build across grades with no repetition. The Children's Math Worlds program was also focused and coherent with core grade-level topics chosen ambitiously to match those of high-achieving countries. However, Math Expressions needed to meet the goals of all states, so some topics such as fractions, geometry, and measurement do repeat across grades, due to the high degree of variation between states. To transition to CCSS, users of Math Expressions from © 2011 or earlier can focus just on the topics that are at the CCSS grade levels. The reasoning, representations, and problem solving in the CCSS are already in those Math Expressions lessons.

LEARNING PATH TEACHING-LEARNING

Even before CCSS, Math Expressions used a Learning Path Teaching-Learning Model that enables students to develop understanding and fluency (see Figure 1). This model is based on

Dr. Karen Fuson, Program Author of Math Expressions: Common Core and Professor Emerita of Learning Sciences, School of Education and Social Policy, Northwestern University
National Research Council Reports and on the NCTM Process Standards. Each new math topic begins with Phase 1 in which teachers elicit and the class discusses student methods. The focus here is on understanding, and many students are able to develop methods because *Math Expressions* has carefully developed prerequisite knowledge. Phase 2 begins soon after this. Methods gathered in the extensive classroom research that developed *Math Expressions* are introduced in lessons. These methods are mathematically desirable and deserving of time and discussion in the class. These are formal math methods, but have more accessible forms of recording what can be unnecessarily difficult methods. In Phase 3, students use these methods in the middle without drawings as the methods become fluent.

The learning path through all three phases for a given topic allows a whole progression of methods to be enacted, discussed, and related. Because all students can share a method, and all students see the whole learning path of methods, this approach supports differentiating within whole-class activities. The crucial feature is to help students move on as soon as possible to a mathematically desirable, advanced method in the middle that the student can understand and explain. The learning paths within and across grades in the CCSS contain similar relationships between understanding and fluency.

### THE CCSS MATHEMATICAL PRACTICES

The Common Core Mathematical Practices are crucial for supporting the movement among the phases (see the arrows in Figure 1). There are eight Mathematical Practices, but these are organized into four pairs that are easier to remember and use (see Figure 2). These Mathematical Practices are carried out within a Math Talk Community; Math Talk is a major support for the teaching and learning in the classroom community.

Students and teacher continually engage in Math Sense-Making about Math Structure. They show such structure in Math Drawings that support Math Explaining instructional conversations in which methods are explained, questioned, and justified. Math Drawings allow students to relate steps in a situation or a computation to parts of the drawing and help all students understand each other’s explanations. The classroom actions summarized in the Common Core Mathematical Practices have always been part of *Math Expressions* classrooms. However, the four pairs that can summarize the Common Core Mathematical Practices are helpful verbalizations about central foci of teaching *Math Expressions* within the Math Talk Community.

Teachers can share how they are using and deepening Math Sense-Making about Math Structure with Math Explaining supported by Math Drawings in their classrooms every day.

---

**Figure 1**

National Research Council Reports and on the NCTM Process Standards. Each new math topic begins with Phase 1 in which teachers elicit and the class discusses student methods. The focus here is on understanding, and many students are able to develop methods because *Math Expressions* has carefully developed prerequisite knowledge. Phase 2 begins soon after this. Methods gathered in the extensive classroom research that developed *Math Expressions* are introduced in lessons. These methods are mathematically desirable and deserving of time and discussion in the class. These are formal math methods, but have more accessible forms of recording what can be unnecessarily difficult methods. In Phase 3, students use these methods in the middle without drawings as the methods become fluent.

The learning path through all three phases for a given topic allows a whole progression of methods to be enacted, discussed, and related. Because all students can share a method, and all students see the whole learning path of methods, this approach supports differentiating within whole-class activities. The crucial feature is to help students move on as soon as possible to a mathematically desirable, advanced method in the middle that the student can understand and explain. The learning paths within and across grades in the CCSS contain similar relationships between understanding and fluency.

### THE CCSS MATHEMATICAL PRACTICES

The Common Core Mathematical Practices are crucial for supporting the movement among the phases (see the arrows in Figure 1). There are eight Mathematical Practices, but these are organized into four pairs that are easier to remember and use (see Figure 2). These Mathematical Practices are carried out within a Math Talk Community; Math Talk is a major support for the teaching and learning in the classroom community.

Students and teacher continually engage in Math Sense-Making about Math Structure. They show such structure in Math Drawings that support Math Explaining instructional conversations in which methods are explained, questioned, and justified. Math Drawings allow students to relate steps in a situation or a computation to parts of the drawing and help all students understand each other’s explanations. The classroom actions summarized in the Common Core Mathematical Practices have always been part of *Math Expressions* classrooms. However, the four pairs that can summarize the Common Core Mathematical Practices are helpful verbalizations about central foci of teaching *Math Expressions* within the Math Talk Community.

Teachers can share how they are using and deepening Math Sense-Making about Math Structure with Math Explaining supported by Math Drawings in their classrooms every day.
**RESPONSIVE TEACHING**

Teachers and students in a Math Talk Community use responsive means of assistance when using the CCSS Mathematical Practices. The three main categories create the Math Talk Community as everyone engages and involves others, manages materials and themselves, and coaches. Coaching includes five supportive interactions that the teacher and students use to help everyone focus on and connect the CCSS Mathematical Practices shown in Figure 2.

**Responsive Means of Assisting Learning**

<table>
<thead>
<tr>
<th>Engaging and Involving</th>
<th>Managing</th>
<th>Coaching</th>
</tr>
</thead>
<tbody>
<tr>
<td>includes:</td>
<td>includes helping students</td>
<td>includes five supportive interactions that guide student learning:</td>
</tr>
<tr>
<td>• inviting all students to share ideas and questions</td>
<td>• monitor</td>
<td><strong>Model</strong></td>
</tr>
<tr>
<td>• promoting analysis and discussion</td>
<td>• be responsible for</td>
<td><strong>Cognitive Structure and Clarify</strong></td>
</tr>
<tr>
<td>• expecting that all students participate in developing understanding together in the community.</td>
<td>• take ownership of</td>
<td><strong>Instruct/Explain</strong></td>
</tr>
<tr>
<td></td>
<td>their own learning.</td>
<td><strong>Question</strong></td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>Give Feedback</strong></td>
</tr>
</tbody>
</table>

Student Leaders help with all aspects of the Math Talk Community and can assist individuals when working problems. Student Leaders also lead Quick Practice routines; these build needed skills and understandings. Quick Practice should be done every day.

Core aspects of each CCSS math domain are now briefly outlined. These core aspects all are consistent with the approaches of *Math Expressions*. Check for more detailed author papers on the domains at hmheducation.com/mathexpressions

---

**Common Core Mathematical Practices Used in a Math Talk Community**

<table>
<thead>
<tr>
<th>Math Sense-Making: Make sense and use appropriate precision</th>
<th>Math Drawings: Model and use tools</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Make sense of problems and persevere in solving them.</td>
<td>4 Model with mathematics.</td>
</tr>
<tr>
<td>6 Attend to precision.</td>
<td>5 Use appropriate tools strategically.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Math Structure: See structure and generalize</th>
<th>Math Explaining: Reason, explain, and question</th>
</tr>
</thead>
<tbody>
<tr>
<td>7 Look for and make use of structure.</td>
<td>2 Reason abstractly and quantitatively.</td>
</tr>
<tr>
<td>8 Look for and express regularity in repeated reasoning.</td>
<td>3 Construct viable arguments and critique the reasoning of others.</td>
</tr>
</tbody>
</table>

---

Figure 2

**Figure 3**
**OA: OPERATIONS AND ALGEBRAIC THINKING**

This domain lays out an ambitious learning path with word-problem types as bases for understanding of operations (+ - x ÷). All problem types are used with all three unknowns so that students have experience with the algebraic problems from Grade 1 on.

*Math Expressions* has always used this full range of research-based situations (see the author paper on OA and problem solving for more details).

The CCSS specify that students are to work with many forms of equations to build algebraic understanding. *Math Expressions* does this. For example, equations such as $5 = 2 + 3$ and $5 = 4 + 1$ are the first equation types Kindergarten and Grade 1 children see. These equations record each decomposition of a number into two addends. Adding is recorded by equations such as $2 + 3 = 5$.

The CCSS specify a learning path of levels of addition/subtraction strategies from Kindergarten through Grade 2: Level 1 direct model, Level 2 count on, and Level 3 recomposing, such as make a ten and other derived fact methods. Students also move through a learning path for multiplication/division strategies. Central to these single-digit operations are understanding subtraction as an unknown-addend problem and division as an unknown-factor problem. *Math Expressions* provides explicit and deep support for these learning paths.

---

**Example Problem:**

7 children were on the swings. Some were on the slides. 15 were playing altogether. How many children were on the slides?

$$7 + \square = 15$$

**Diagram:**

```
swings  slides  total
playing
15
7  \square
```

---

**Problem:**

To prepare for a family gathering, Sara and Ryan made soup.

Sara made 6 quarts. Ryan made 18 quarts.

Ryan made how many times as many quarts as Sara made?

**Solution:**

Ryan made 3 times as many quarts as Sara.  
$$r = 3 \cdot s \text{ or } r = 3s$$

<table>
<thead>
<tr>
<th>Sara (s)</th>
<th>Ryan (r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>6 6 6 18</td>
</tr>
<tr>
<td>6 6 6 6</td>
<td>6</td>
</tr>
</tbody>
</table>
NBT: NUMBERS AND OPERATIONS IN BASE TEN

For multi-digit computation in the CCSS, students are to develop, discuss, and use efficient, accurate, and general methods, including the standard algorithm. They initially use concrete models or drawings and strategies based on place value and properties of operations, and they relate the strategy to a written method and explain the reasoning used (explanations may be supported by drawings or objects).

Math Expressions has always approached multi-digit computation in these ways. The MathBoards provide supports for students to learn to make math drawings to show quantities. Students draw hundreds, tens, and ones in Grades 1, 2, and 3, and thousands and ten-thousands in Grade 4. Math Expressions uses and generalizes array/area models in Grade 3 for single-digit multiplication and division. These are extended to multi-digit multiplication and division in Grades 4 and 5.

Math Expressions uses Secret Code Cards to show the meaning of place values for whole numbers and for decimals. These cards and the math drawings help students explain their methods using concepts of place value, as specified in the CCSS.

---

**Practice Addition**
Add. Use any method.

86
+ 57  
\[ \frac{86}{130} \quad \text{or} \quad \frac{57}{143} \]
+ 13  
143

130 + 13 = 143

---

**Expanded Notation Method**

\[
\begin{array}{c}
67 = 60 + 7 \\
40 \times 60 = 2,400 \\
40 \times 7 = 280 \\
3 \times 60 = 180 \\
3 \times 7 = 21
\end{array}
\]

**Transitional Method**

\[
\begin{align*}
67 & = 60 + 7 \\
40 \times 60 & = 2,400 \\
40 \times 7 & = 280 \\
3 \times 60 & = 180 \\
3 \times 7 & = 21
\end{align*}
\]

**Short Method**

\[
\begin{align*}
67 & = 60 + 7 \\
40 \times 60 & = 2,400 \\
40 \times 7 & = 280 \\
3 \times 60 & = 180 \\
3 \times 7 & = 21
\end{align*}
\]

Two thousand, four hundred thirty five
2 thousands, 4 hundreds, 3 tens, 5 ones

\[
\begin{array}{c}
2,000 \quad 400 \quad 30 \quad 5
\end{array}
\]

2,000 + 400 + 30 + 5
NF: NUMBER AND OPERATIONS—FRACTIONS

In the CCSS, fractions are made by composing unit fractions 1/n. Each unit fraction is the result of dividing some whole into n equal parts. Operations with, and comparisons of, fractions use unit fractions and drawings that show unit fractions, and students are to understand and explain such operations and comparisons. Fractions of sets involve fraction multiplication, rather than being part of the initial concepts of fractions. Division of fractions is related to multiplication of fractions. Math Expressions already used all of these conceptual features of fractions before CCSS. The approaches are deeply consistent.

To understand fractions, students fold fraction strips and see and label bar drawings. They discuss and generalize the unit structure as they make more parts of the same whole: the unit fraction becomes smaller as the denominator becomes larger. To help students understand number line diagrams, students loop lengths and see bar drawings beside the number line diagram, as shown on the right. Students see drawings of fractions to support their explaining the math structure in equivalent fractions: these have more but smaller parts or fewer but larger parts.

Students see chains of equivalent fractions as two rows from the multiplication table to emphasize the many equivalent fractions that are possible.

Writing fractions as sums of unit fractions (see top right) helps to overcome a typical error when adding fractions: adding the denominators as well as adding the numerators for like (or later, unlike) unit fractions. Students analyze cases for choosing common denominators to add, subtract or compare unlike fractions. They use length and area models to find products. For example, students find 2/3 of each of the 4/5 and explain the general structure: multiply the numerators and multiply the denominators. Dividing begins with cases where one can divide numerators and denominators. Students then tackle cases where such dividing is not possible. They unsimplify the dividend to be able to divide, then divide and see the structure: Dividing by a/b is the same as multiplying by b/a.

Continual use of the Mathematical Practices to visualize all operations with fractions helps students overcome errors they make by over-generalizing from whole numbers.
**MD: MEASUREMENT AND DATA AND G: GEOMETRY**

In the CCSS, working with data is integrated with problem solving in the early grades as students solve the whole range of word problems using data presented in graphs. Work with data presentation formats is focused and builds slowly through the CCSS learning path across grade levels. At Grade 6, students do significant in-depth work with statistics and probability.

Understanding concepts of measurement units for different kinds of measure builds from considering measurable attributes in Kindergarten to concentrations on length in Grades 1 and 2, on area of rectangles and perimeter in Grade 3, on angles in Grade 4, on volume in Grade 5, and on area of many shapes and surfaces in Grade 6. Other measures such as time, money, liquid volumes, and masses begin with some measures in one grade and extend to more measures and conversions of measures in later grades.

Geometry is focused and more coherent than across the many state standards. A major learning path involves composing/decomposing, analyzing, and relating 2D shapes, culminating in Grade 5 in classifying in a hierarchy of categories based on properties of the shapes. Students also analyze, discuss, and compose 3D shapes. Partitioning the simplest shapes into equal shares leads into concepts of unit fractions. Coordinate graphing begins in Grade 5 and builds to four quadrants in Grade 6.

**Math Expressions** has systematic work along these learning paths in all domains. Picture and bar graphs are related to comparison situations and comparison drawings. Students pose all types of questions about data forms, thereby extending student work in the QA domain. In-depth work with units of length and how a ruler is built up from units sets the stage for working with other kinds of units. Rulers, number-line diagrams, and graphing scales are related to each other to create shared contexts across different mathematical domains. Emphasizing the nature of the unit for each kind of measure helps students distinguish and explain perimeter, area, volume, and surface area.

Ask and answer math questions about the animals.

![Read a Horizontal Bar Graph](chart)

Use the bar graph to complete the sentences. Write the number. Ring more or fewer.

1. There are [ ] more / fewer tigers than bears.
2. There are [ ] more / fewer monkeys than tigers.

Students see the meaning of length units and how units are composed to make a ruler.

![Understanding classification of shapes requires understanding that shared attributes can define a larger category.](diagram)

Understanding classification of shapes requires understanding that shared attributes can define a larger category.
REFERENCES


