Anticipation Guide

Conic Sections

Step 1

Before you begin Chapter 10

• Read each statement.
• Decide whether you Agree (A) or Disagree (D) with the statement.
• Write A or D in the first column OR if you are not sure whether you agree or disagree, write NS (Not Sure).

<table>
<thead>
<tr>
<th>STEP 1</th>
<th>Statement</th>
<th>STEP 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>A, D, or NS</td>
<td>Statement</td>
<td>A or D</td>
</tr>
<tr>
<td>1.</td>
<td>To find the midpoint between two points on a coordinate plane, find the means of the coordinates of the points.</td>
<td>A</td>
</tr>
<tr>
<td>2.</td>
<td>All points on a parabola are the same distance from a given point and the x-axis.</td>
<td>D</td>
</tr>
<tr>
<td>3.</td>
<td>In the equation for a parabola, $y = a(x - h)^2 + k$, if $a &lt; 0$ the parabola opens downward.</td>
<td>A</td>
</tr>
<tr>
<td>4.</td>
<td>If the equation of a circle is $(x - h)^2 + (y - k)^2 = r^2$, then the center of the circle is at the point $(h, k)$.</td>
<td>D</td>
</tr>
<tr>
<td>5.</td>
<td>A tangent line to a circle is a line that intersects the circle in two points.</td>
<td>D</td>
</tr>
<tr>
<td>6.</td>
<td>The foci of an ellipse always lie on the major axis.</td>
<td>A</td>
</tr>
<tr>
<td>7.</td>
<td>The asymptotes of a graph of a hyperbola are lines that the graph approaches but never reaches.</td>
<td>A</td>
</tr>
<tr>
<td>8.</td>
<td>Conic sections are formed by the intersection of two cones.</td>
<td>D</td>
</tr>
<tr>
<td>9.</td>
<td>When the equation of a conic section is written in standard form it is possible to name which conic section the equation represents.</td>
<td>A</td>
</tr>
<tr>
<td>10.</td>
<td>No solution can be found for a system of equations containing both linear and quadratic equations.</td>
<td>D</td>
</tr>
</tbody>
</table>

Step 2

After you complete Chapter 10

• Reread each statement and complete the last column by entering an A or a D.
• Did any of your opinions about the statements change from the first column?
• For those statements that you mark with a D, use a piece of paper to write an example of why you disagree.
10-1 Study Guide and Intervention
Midpoint and Distance Formulas

The Midpoint Formula
The midpoint M of a segment with endpoints \((x_1, y_1)\) and \((x_2, y_2)\) is 
\[
M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
\]

**Example 1** Find the midpoint of the line segment with endpoints at \((4, -7)\) and \((-2, 3)\).
\[
M_{12} = \left( \frac{4 + (-2)}{2}, \frac{-7 + 3}{2} \right)
\]
\[
= \left( \frac{2}{2}, \frac{-4}{2} \right) = (1, -2)
\]
The midpoint of the segment is \((1, -2)\).

**Example 2** A diameter \(AB\) of a circle has endpoints \(A(5, -11)\) and \(B(-7, 6)\). What are the coordinates of the center of the circle?
The center of the circle is the midpoint of all of its diameters.
\[
M_{12} = \left( \frac{5 + (-7)}{2}, \frac{-11 + 6}{2} \right)
\]
\[
= \left( \frac{-2}{2}, -\frac{5}{2} \right) = (-1, -\frac{5}{2})
\]
The circle has center \((-1, -\frac{5}{2})\).

**Exercises**
Find the midpoint of each line segment with endpoints at the given coordinates.
1. \((12, 7)\) and \((-2, 11)\)
2. \((-8, -3)\) and \((10, 9)\)
3. \((4, 15)\) and \((10, 1)\)
4. \((-3, -3)\) and \((3, 3)\)
5. \((15, 6)\) and \((12, 14)\)
6. \((-2, -8)\) and \((-10, \text{6})\)
7. \((0, 0)\) and \((13, 5, 10)\)
8. \(\left(\frac{-3}{2}, 8\right)\)
9. \(\left(\frac{-1}{2}, -1\right)\)
10. \((-7, -6)\) and \((-1, 24)\)
11. \((3, -10)\) and \((30, -20)\)
12. \((-9, 1.7)\) and \((-11, 1.3)\)
13. Segment \(MN\) has midpoint \(P\). If \(M\) has coordinates \((-4, 3)\) and \(P\) has coordinates \((-8, 6)\), what are the coordinates of \(N\)?
14. Circle \(M\) has a diameter \(ST\). If \(M\) has coordinates \((-4, -8)\) and \(P\) has coordinates \((2, -10)\), what are the coordinates of \(T\)?
15. Segment \(AB\) has midpoint \(C\). \(A\) has coordinates \((-5, 4)\) and \(C\) has coordinates \((10, 11)\). What are the coordinates of \(B\) and \(D\)?

The Distance Formula
The distance between two points \((x_1, y_1)\) and \((x_2, y_2)\) is given by
\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

**Example 1** What is the distance between \((8, -2)\) and \((-6, -8)\)?
\[
d = \sqrt{(-6 - 8)^2 + (-2 - (-8))^2}
\]
\[
= \sqrt{-14^2 + 6^2}
\]
\[
= \sqrt{196 + 36}
\]
\[
= 232
\]

**Example 2** Find the perimeter and area of square \(PQRS\) with vertices \(P(-4, 1), Q(-2, 7), R(4, 5),\) and \(S(2, -1)\).
Find the length of one side to find the perimeter and the area. Choose \(PQ\).
\[
d = \sqrt{(-4 - (-2))^2 + (1 - 7)^2}
\]
\[
= \sqrt{(-2)^2 + (-6)^2}
\]
\[
= \sqrt{10}
\]

Since one side of the square is \(10\) units, the area is \(100\) units².

**Exercises**
Find the distance between each pair of points with the given coordinates.
1. \((3, 7)\) and \((-1, 4)\)
2. \((-2, -10)\) and \((10, -5)\)
3. \((6, -6)\) and \((-2, 0)\)
4. \((7, 2)\) and \((4, -1)\)
5. \((-5, -2)\) and \((3, 4)\)
6. \((11, 5)\) and \((16, 9)\)
7. \((-3, 4)\) and \((6, -11)\)
8. \((13, 9)\) and \((11, 15)\)
9. \((-5, -7)\) and \((2, 12)\)

10. Rectangle \(ABCD\) has vertices \(A(1, 4), B(3, 11), C(-3, -2),\) and \(D(-5, 1)\). Find the perimeter and area of \(ABCD\). \(2\sqrt{13} + 6\sqrt{5}\) units; \(3\sqrt{65}\) units²

11. Circle \(R\) has diameter \(ST\) with endpoints \(S(4, 5)\) and \(T(-2, -3)\). What are the circumference and area of the circle? (Express your answer in terms of \(\pi\)).

10\pi\) units; \(25\pi\) units²
Find the midpoint of each line segment with endpoints at the given coordinates.

1. (4, -1), (-4, 1)  
   \((0, 0)\)

2. (-1, 4), (5, 2)  
   \((2, 3)\)

3. (3, 4), (5, 4)  
   \((4, 4)\)

4. (6, 2), (2, -1)  
   \((4, 1)\)

5. (3, 9), (-2, -3)  
   \((-\frac{1}{2}, 3)\)

6. (-3, 5), (-3, -8)  
   \((-3, -\frac{3}{2})\)

7. (3, 2), (-5, 0)  
   \((-1, 1)\)

8. (3, -4), (5, 2)  
   \((4, -1)\)

9. (-5, -9), (5, 4)  
   \((0, -\frac{5}{2})\)

10. (-11, 14), (0, 4)  
    \((-\frac{11}{2}, 9)\)

11. (3, -6), (-8, -3)  
    \((-\frac{5}{2}, -\frac{9}{2})\)

12. (0, 10), (-2, -5)  
    \((-1, \frac{5}{2})\)

Find the distance between each pair of points with the given coordinates.

13. (4, 12), (-1, 0)  
    13 units

14. (7, 7), (-5, -2)  
    15 units

15. (-1, 4), (1, 4)  
    2 units

16. (11, 11), (8, 15)  
    5 units

17. (1, -6), (7, 2)  
    10 units

18. (3, -5), (3, 4)  
    9 units

19. (2, 3), (3, 5)  
    \(\sqrt{5}\) units

20. (-4, 3), (-1, 7)  
    5 units

21. (-5, -5), (3, 10)  
    17 units

22. (3, 9), (-2, -3)  
    13 units

23. (6, -2), (-1, 3)  
    \(\sqrt{74}\) units

24. (-4, 1), (2, -4)  
    \(\sqrt{61}\) units

25. (0, -3), (4, 1)  
    \(4\sqrt{2}\) units

26. (-5, -6), (2, 0)  
    \(\sqrt{85}\) units

Find the midpoint of each line segment with endpoints at the given coordinates.

1. (8, -3), (-6, -11)  
   \((1, -7)\)

2. (-14, 5), (10, 6)  
   \((-2, \frac{11}{2})\)

3. (-7, -6), (1, -2)  
   \((-3, -4)\)

4. (8, -2), (8, -8)  
   \((8, -5)\)

5. (9, -4), (1, -1)  
   \((\frac{5}{2}, \frac{9}{2})\)

6. (3, 3), (4, 9)  
   \((\frac{7}{2}, 6)\)

7. (4, -2), (3, -7)  
   \((\frac{7}{2}, -\frac{9}{2})\)

8. (6, 7), (4, 4)  
   \((5, \frac{11}{2})\)

9. (-4, -2), (-8, 2)  
   \((-6, 0)\)

10. (5, -2), (3, 7)  
    \((4, \frac{5}{2})\)

11. (-6, 3), (-5, -7)  
    \((-\frac{11}{2}, -2)\)

12. (-9, -8), (8, 3)  
    \((-\frac{1}{2}, -\frac{5}{2})\)

13. (2, 6), (-4, 7), (8, 4, 2, 5)  
    \((5.5, -1.1)\)

14. \(\left(\frac{1}{3}, \frac{6}{3}, \frac{2}{3}, \frac{4}{3}\right)\)  
    \((\frac{1}{6}, 5)\)

15. (-2.5, -4.2), (8, 1, 4, 2)  
    \((2.8, 0)\)

Find the distance between each pair of points with the given coordinates.

17. (5, 2), (2, -2)  
    5 units

18. (-2, -4), (4, 4)  
    10 units

19. (-3, 8), (-1, -5)  
    \(\sqrt{173}\) units

20. (0, 1), (9, -6)  
    \(\sqrt{130}\) units

21. (-5, 6), (-6, 6)  
    1 unit

22. (-3, 5), (12, -3)  
    17 units

23. (-2, -3), (9, 3)  
    \(\sqrt{157}\) units

24. (-9, -8), (-7, 8)  
    \(2\sqrt{65}\) units

25. (9, 3), (9, -2)  
    5 units

26. (-1, -7), (0, 6)  
    \(\sqrt{170}\) units

27. (10, -3), (-2, -8)  
    13 units

28. (-0.5, -6), (1, 5, 0)  
    \(2\sqrt{10}\) units

29. \(\left(\frac{2}{5}, \frac{3}{5}, \frac{1}{5}\right)\)  
    1 unit

30. \((-4\sqrt{2}, -\sqrt{5}), (\sqrt{2}, 4\sqrt{5})\)  
    \(\sqrt{127}\) units

31. GEOMETRY  
   Circle O has a diameter AB. If A is at (-6, -2) and B is at (-3, 4), find the center of the circle and the length of its diameter.  
   \((-\frac{9}{2}, 1)\)  
   \(3\sqrt{5}\) units

32. GEOMETRY  
   Find the perimeter of a triangle with vertices at (-1, 3), (-4, 9), and (-2, 1).  
   \(18 + 2\sqrt{17}\) units
1. **EXHIBITS** Museum planners want to place a statue directly in the center of their Special Exhibits Room. Suppose the room is placed on a coordinate plane as shown. What are the coordinates of the center of this room?

2. **WALKING** Laura starts at the origin. She walks 8 units to the right and then 12 units up. How far away from the origin is she? Round your answer to the nearest tenth.

   14.4 units

3. **SURVEILLANCE** A grid is superimposed on a map of Texas. Dallas has coordinates (200, 5) and Amarillo has coordinates (−100, 208). If each unit represents 1 mile, how long will it take a plane flying at an average speed of 410 miles per hour to fly directly from Dallas to Amarillo? Round your answer to the nearest tenth of an hour.

   0.9 hour

4. **AIRPLANES** A grid is superimposed on a map of Texas. Dallas has coordinates (200, 5) and Amarillo has coordinates (−100, 208). If each unit represents 1 mile, how long will it take a plane flying at an average speed of 410 miles per hour to fly directly from Dallas to Amarillo? Round your answer to the nearest tenth of an hour.

   0.9 hour

5. **TRAVEL** For Exercises 5 and 6, use the following information and the figure below.

   The Martinez family is planning a trip from their home in Fort Lauderdale to Tallahassee. They plan to stop overnight at a location about halfway between the two cities.

6. **How many miles is it from Fort Lauderdale to Tallahassee? Round your answer to the nearest mile.**

   521 mi

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**Distance Between Points in Space**

The Distance Formula and Midpoint Formula on the coordinate plane is derived from the Pythagorean Theorem $a^2 + b^2 = c^2$.

In three dimensions, the coordinate grid contains the $x$-axis and the $y$-axis, as in two-dimensional geometry, and also a $z$-axis. An example of a line segment drawn on a three-dimensional coordinate grid is shown at the right.

The three-dimensional distance formula is much like the one for two dimensions. The distance from $(x_1, y_1, z_1)$ to $(x_2, y_2, z_2)$ can be found using $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

**Example**

Find the distance between the points $(1, 3, 0)$ and $(0, 1, 3)$.

$d = \sqrt{(1 - 0)^2 + (3 - 1)^2 + (0 - 3)^2}$

Replace $(x_1, y_1, z_1)$ with $(1, 3, 0)$ and $(x_2, y_2, z_2)$ with $(0, 1, 3)$.

$d = \sqrt{1 + 4 + 9}$ or $\sqrt{14}$

Simplify.

**Exercises**

1. Find the distance between each pair of points $A(1, 3, -2)$ and $B(4, 2, 1)$.

   $d = \sqrt{19}$

2. Find the distance between each pair of points $C(5, 3, 2)$ and $D(6, 1, 7)$.

   $d = 5\sqrt{2}$

3. Find the distance between each pair of points $E(-2, -1, 6)$ and $F(-1, 3, 2)$.

   $d = 3\sqrt{10}$

4. Use what you know about the midpoint formula for a segment graphed on a regular coordinate grid to make a conjecture about the formula for finding the coordinates of a midpoint in three-dimensions.

   Midpoint $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

5. Find the midpoint for each segment in Exercises 1-3.

   $(5, 2, -1, 2), (5, 2, 2, 9), (-3, 2, 1)$
Parabolas

Get Ready for the Lesson

Read the introduction to Lesson 10-2 in your textbook.

Name at least two reflective objects that might have the shape of a parabola.

Sample answer: telescope mirror, satellite dish

Read the Lesson

1. In the parabola shown in the graph, the point (2, -2) is called the ______ vertex ______ and the point (2, 0) is called the ______ focus ______. The line y = -4 is called the ______ directrix ______, and the line x = 2 is called the ______ axis of symmetry ______.

2. a. Write the standard form of the equation of a parabola that opens upward or downward. y = a(x - h)² + k
   b. The parabola opens downward if ______ a < 0 ______ and opens upward if ______ a > 0 ______. The equation of the axis of symmetry is ______ x = h ______, and the coordinates of the vertex are ______ (h, k) ______.

3. A parabola has equation x = -1/8(y - 2)² + 4. This parabola opens to the ______ left ______. It has vertex ______ (4, 2) ______ and focus ______ (2, 2) ______. The directrix is ______ x = 6 ______. The length of the latus rectum is ______ 8 ______ units.

Remember What You Learned

4. How can the way in which you plot points in a rectangular coordinate system help you to remember what the sign of a tells you about the direction in which a parabola opens?
   Sample answer: In plotting points, a positive x-coordinate tells you to move to the right and a negative x-coordinate tells you to move to the left. This is like a parabola whose equation is of the form "x = ..."; it opens to the right if a > 0 and to the left if a < 0. Likewise, a positive y-coordinate tells you to move up and a negative y-coordinate tells you to move down. This is like a parabola whose equation is of the form "y = ..."; it opens upward if a > 0 and downward if a < 0.

Example

Identify the coordinates of the vertex and focus, the equations of the axis of symmetry and directrix, and the direction of opening of the parabola with equation y = 2x² - 12x - 25.

- Original equation: y = 2x² - 12x - 25
- Factor 2 from the x-terms: y = 2(x² - 6x) - 25
- Complete the square on the right side: y = 2(x² - 6x + 9) - 25 - 2(9)
- Write in standard form: y = 2(x - 3)² - 43

The vertex of this parabola is located at (3, -43), the focus is located at (3, -43 - 4), the equation of the axis of symmetry is x = 3, and the equation of the directrix is y = -43 + 1/2.

The parabola opens upward.

Exercises

Identify the coordinates of the vertex and focus, the equations of the axis of symmetry and directrix, and the direction of opening of the parabola with the given equation.

1. y = x² + 6x - 4
2. y = 8x - 4x² + 10
3. y = x² - 8x + 6
   (-3, -13), (2, 18), (1, 17)
   (-10, 4), (-9, 4), (-4, 1)
   (-3, -12), x = -3
   x = 2, y = 18, y = x = -10
   y = -13, up, down
   right

Write an equation of each parabola described below.

4. Focus (-2, 3), directrix x = -2
5. Vertex (5, 1), focus (5, 1/2)

x = 6(y - 3)² - 2
x = -3(y - 1)² + 5
Study Guide and Intervention (continued)

Parabolas

Graph Parabolas To graph an equation for a parabola, first put the given equation in standard form.

\[ y = a(x - h)^2 + k \] for a parabola opening up or down, or \[ x = a(y - k)^2 + h \] for a parabola opening to the left or right

Use the values of \( a, h, \) and \( k \) to determine the vertex, focus, axis of symmetry, and length of the latus rectum. The vertex and the endpoints of the latus rectum give three points on the parabola. If you need more points to plot an accurate graph, substitute values for points near the vertex.

Example

Graph \( y = \frac{1}{3}(x - 1)^2 + 2 \).

In the equation, \( a = \frac{1}{3}, h = 1, k = 2 \).

The parabola opens up, since \( a > 0 \).

vertex: \((1, 2)\)
axis of symmetry: \( x = 1 \)
focus: \( \left(1, 2 + \frac{1}{4}\right) \) or \( \left(1, \frac{9}{4}\right) \)
length of latus rectum: \( \frac{1}{\frac{1}{3}} \) or 3 units
endpoints of latus rectum: \( \left(2\frac{1}{2}, \frac{9}{4}\right) \) and \( \left(-\frac{1}{2}, \frac{9}{4}\right) \)

Exercises

The coordinates of the focus and the equation of the directrix of a parabola are given. Write an equation for each parabola and draw its graph.

1. \( (1, 2) \); vertex: \((1, 2)\)
axis of symmetry: \( x = 1 \)
focus: \( \left(1, 2 + \frac{1}{4}\right) \) or \( \left(1, \frac{9}{4}\right) \)
length of latus rectum: \( \frac{1}{\frac{1}{3}} \) or 3 units
endpoints of latus rectum: \( \left(2\frac{1}{2}, \frac{9}{4}\right) \) and \( \left(-\frac{1}{2}, \frac{9}{4}\right) \)

Write an equation for each parabola described below. Then draw the graph.

1. \( y = 3(x - 3)^2 + 3 \)
2. \( y = \frac{1}{4}(x - 4)^2 - 5 \)
3. \( x = \frac{1}{4}(y + 1)^2 + 4 \)
4. \( y = (x - 2)^2 \)
5. \( x = (y - 2)^2 + 3 \)
6. \( y = -(x + 3)^2 + 4 \)
7. \( y = -3x^2 \)
8. \( y = (x - 5)^2 + 1 \)
9. \( x = 2(y - 3)^2 + 1 \)
Write each equation in standard form.

1. \( y = 2x^2 - 12x + 19 \)
2. \( y = \frac{1}{2}x^2 + 3x + \frac{1}{2} \)
3. \( y = -3x^2 - 12x - 7 \)
4. \( y = 2(x - 3)^2 + 1 \)
5. \( y = \frac{1}{2}(x - (-3))^2 + (-4) \)
6. \( y = -3(x - (-2))^2 + 5 \)

Identify the coordinates of the vertex and focus, the equations of the axis of symmetry and directrix, and the direction of opening of the parabola with the given equation. Then find the length of the latus rectum and graph the parabola.

4. \( y = (x - 4)^2 + 3 \)
5. \( x = -\frac{1}{3}y^2 + 1 \)
6. \( x = 3(y + 1)^2 - 3 \)

Write an equation for each parabola described below. Then draw the graph.

7. vertex (0, -4), focus (0, -\frac{7}{8})
8. vertex (-2, 1), directrix \( x = -3 \)
9. vertex (1, 3), axis of symmetry \( x = 1 \), latus rectum 2 units, \( a < 0 \)
10. TELEVISION Write the equation in the form \( y = ax^2 \) for a satellite dish. Assume that the bottom of the upward-facing dish passes through (0, 0) and that the distance from the bottom to the focus point is 8 inches. \( y = \frac{1}{32}x^2 \)

1. PROJECTILE A projectile follows the path of the parabola \( y = -\frac{3}{2}x^2 + 6x \). Sketch the path of the projectile by graphing the parabola.

2. COMMUNICATION David has just made a large parabolic dish whose cross section is based on the graph of the parabola \( y = 0.25x^2 \). Each unit represents one foot and the diameter of his dish is 4 feet. He wants to make a listening device by placing a microphone at the focus of the parabola. Where should the microphone be placed?

3. BRIDGES A bridge is in the shape of a parabola that opens downward. The equation of the parabola to model the arch of the bridge is given by \( y = \frac{x^2}{24} + \frac{5}{6}x + \frac{11}{6} \) where each unit is equivalent to 1 yard. The \( x \)-axis is the ground level. What is the maximum height of the bridge above the ground?

6 yd

4. TELESCOPES An astronomer is working with a large reflecting telescope. The reflecting mirror in the telescope has the parabolic cross section shown in the graph whose equation is given by \( y = \frac{1}{8}(x - 4)^2 + 2 \). Each unit represents 1 meter. The astronomer is standing at the origin. How far from the focus of the parabola is the point on the mirror directly over the astronomer’s head?

4 m

5. Write the equation of the parabola to model the arch.

\( y = -\frac{1}{125}x^2 + 10 \)

6. Identify the coordinates of the focus of this parabola.

(0, -21.25)
Limits

Sequences of numbers with a rational expression for the general term often approach some number as a finite limit. For example, the reciprocals of the positive integers approach 0 as \( n \) gets larger and larger. This is written using the notation shown below. The symbol \( \lim \) stands for infinity and \( n \to \infty \) means that \( n \) is getting larger and larger, or "\( n \) goes to infinity."

\[
\begin{align*}
1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots & \quad \lim_{n \to \infty} \frac{1}{n} = 0 \\
\end{align*}
\]

Example

Find \( \lim_{n \to \infty} \frac{n^2}{n^2 + 1} \).

It is not immediately apparent whether the sequence approaches a limit or not. But notice what happens if we divide the numerator and denominator of the general term by \( n^2 \):

\[
\begin{align*}
\frac{n^2}{n^2 + 1} & = \frac{n^2}{n^2 + 2n + 1} \\
& = \frac{1}{1 + \frac{2n}{n^2} + \frac{1}{n^2}} \\
& = \frac{1}{1 + \frac{2}{n} + \frac{1}{n^2}}
\end{align*}
\]

The two fractions in the denominator will approach a limit of 0 as \( n \) gets very large, so the entire expression approaches a limit of 1.

Exercises

Find the following limits.

1. \( \lim_{n \to \infty} \frac{n^3 + 5n}{n^4 - 6} = 0 \)
2. \( \lim_{n \to \infty} \frac{1-n}{n} = 0 \)
3. \( \lim_{n \to \infty} \frac{2n + 1}{2n + 1} = 1 \)
4. \( \lim_{n \to \infty} \frac{2n + 1}{3n - \frac{2}{3}} = \frac{2}{3} \)

Spreadsheet Activity

Parabolas

You have learned many of the characteristics of parabolas with vertical and horizontal axes of symmetry. The information is summarized in the table at the right. You can use what you know to create a spreadsheet to analyze given equations of parabolas.

<table>
<thead>
<tr>
<th>Form of Equation</th>
<th>Vertex</th>
<th>Axis of Symmetry</th>
<th>Focus</th>
<th>Directrix</th>
<th>Direction of Opening</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = a(x - h)^2 + k )</td>
<td>((h, k))</td>
<td>( x = h )</td>
<td>( h, k + \frac{1}{4a} )</td>
<td>( h - \frac{1}{4a} )</td>
<td>upward if ( a &gt; 0 ), downward if ( a &lt; 0 )</td>
</tr>
<tr>
<td>( x = a(y - k)^2 + h )</td>
<td>((h, k))</td>
<td>( y = k )</td>
<td>( h + \frac{1}{4a}, k )</td>
<td>( h + \frac{1}{4a} )</td>
<td>right if ( a &gt; 0 ), left if ( a &lt; 0 )</td>
</tr>
</tbody>
</table>

The spreadsheet below uses the equation of a parabola in the form \( y = a(x - h)^2 + k \) or \( x = a(y - k)^2 + h \) to find information about the parabola. \( x \) or \( y \) is entered in Column D and the values of \( a, h, \) and \( k \) are entered into Columns A, B, and C respectively.

Exercises

1. Which row represents the equation \( y = 3x^2 + 24x + 50 \)? row 3
2. Write the standard form of the equation represented by row 2.
   \( x = \frac{1}{4} (y + 1)^2 + 3 \)
3. What formula should be used in cell F2? \( 1/ABS(A2) \)
4. Find the vertex, length of latus rectum, axis of symmetry, focus, directrix, and direction of opening of a parabola with equation \( y - 8)^2 = -4(x - 4) \).
   (8, 4); \( y = 4 \); (7, 4); \( x = 9 \); left
Lesson 10-3

Lesson Reading Guide

Circles

Get Ready for the Lesson

Read the introduction to Lesson 10-3 in your textbook.

A large home improvement chain is planning to enter a new metropolitan area and needs to select locations for its stores. Market research has shown that potential customers are willing to travel up to 12 miles to shop at one of their stores. How can circles help the managers decide where to place their store?

Sample answer: A store will draw customers who live inside a circle with center at the store and a radius of 12 miles. The management should select locations for which as many people as possible live within a circle of radius 12 miles around one of the stores.

Read the Lesson

1. a. Write the equation of the circle with center \((h, k)\) and radius \(r\).
   \[(x - h)^2 + (y - k)^2 = r^2\]
   b. Write the equation of the circle with center \((4, -3)\) and radius 5.
   \[(x - 4)^2 + (y + 3)^2 = 25\]
   c. The circle with equation \((x + 8)^2 + y^2 = 121\) has center \((-8, 0)\) and radius 11.
   d. The circle with equation \((x - 10)^2 + (y + 10)^2 = 1\) has center \((10, -10)\) and radius 1.

2. a. In order to find center and radius of the circle with equation \(x^2 + y^2 + 4x - 6y - 3 = 0\), it is necessary to complete the square. Fill in the missing parts of this process.
   
   \[
   \begin{align*}
   x^2 + y^2 + 4x - 6y - 3 &= 0 \\
   x^2 + y^2 + 4x - 6y &= 3 \\
   (x^2 + 4x + 4) + (y^2 - 6y + 9) &= 3 + 4 + 9 \\
   (x + 2)^2 + (y - 3)^2 &= 16
   \end{align*}
   \]
   b. This circle has radius 4 and center at \((-2, 3)\).

Remember What You Learned

3. How can the distance formula help you to remember the equation of a circle?
   Sample answer: Write the distance formula. Replace \((x_1, y_1)\) with \((h, k)\) and \((x_2, y_2)\) with \((x, y)\). Replace \(d\) with \(r\). Square both sides. Now you have the equation of a circle.

10-3 Study Guide and Intervention

Circles

Equations of Circles

The equation of a circle with center \((h, k)\) and radius \(r\) units is \((x - h)^2 + (y - k)^2 = r^2\).

Example: Write an equation for a circle if the endpoints of a diameter are at \((-4, 3)\) and \((6, -3)\).

Use the midpoint formula to find the center of the circle.
\[(h, k) = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)\]
Midpoint formula
\[
\left( \frac{-4 + 6}{2}, \frac{3 + (-3)}{2} \right) = (1, 0)
\]
Simplify.

Use the coordinates of the center and one endpoint of the diameter to find the radius.
\[r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\]
Distance formula
\[
r = \sqrt{(-4 - 1)^2 + (3 - (-3))^2} = \sqrt{5^2 + 6^2} = \sqrt{41}
\]
Simplify.

The radius of the circle is \(\sqrt{41}\), so \(r^2 = 41\).

An equation of the circle is \((x - 1)^2 + (y - 3)^2 = 41\).

Exercises

Write an equation for the circle that satisfies each set of conditions.

1. center \((8, -3)\), radius 6 \((x - 8)^2 + (y + 3)^2 = 36\)
2. center \((5, -6)\), radius 4 \((x - 5)^2 + (y + 6)^2 = 16\)
3. center \((-5, 2)\), passes through \((-9, 6)\) \((x + 5)^2 + (y - 2)^2 = 32\)
4. endpoints of a diameter at \((6, 6)\) and \((10, 12)\) \((x - 8)^2 + (y - 9)^2 = 13\)
5. center \((3, 6)\), tangent to the \(x\)-axis \((x - 3)^2 + (y - 6)^2 = 36\)
6. center \((-4, -7)\), tangent to \(x\) \(-4 + 2\) \((x + 4)^2 + (y + 7)^2 = 36\)
7. center at \((-2, 8)\), tangent to \(y = -4\) \((x + 2)^2 + (y - 8)^2 = 144\)
8. center \((7, 7)\), passes through \((12, 9)\) \((x - 7)^2 + (y - 7)^2 = 29\)
9. endpoints of a diameter are \((-4, -2)\) and \((8, 4)\) \((x - 2)^2 + (y + 1)^2 = 45\)
10. endpoints of a diameter are \((-4, 3)\) and \((6, -8)\) \((x - 1)^2 + (y + 2.5)^2 = 55.25\)
**10-3 Study Guide and Intervention (continued)**

### Circles

**Graph Circles**

To graph a circle, write the given equation in the standard form of the equation of a circle, \((x - h)^2 + (y - k)^2 = r^2\). Plot the center \((h, k)\) of the circle. Then use \(r\) to calculate and plot the four points \((h + r, k)\), \((h - r, k)\), \((h, k + r)\), and \((h, k - r)\), which are all points on the circle. Sketch the circle that goes through those four points.

**Example**

Find the center and radius of the circle whose equation is \(x^2 + 2x + y^2 + 4y = 11\). Then graph the circle.

\[x^2 + 2x + y^2 + 4y = 11\]

\[x^2 + 2x + 1 + y^2 + 4y + 4 = 11 + 1 + 4\]

\[(x + 1)^2 + (y + 2)^2 = 16\]

Therefore, the circle has its center at \((-1, -2)\) and a radius of \(4\). Four points on the circle are \((-3, -2), (-5, -2), (-1, 2),\) and \((-1, -6)\).

**Exercises**

Find the center and radius of the circle with the given equation. Then graph the circle.

1. \((x + 3)^2 + y^2 = 9\)
   - Center: \((-3, 0)\), Radius: 3
   - Graph:

2. \(x^2 + (y + 5)^2 = 4\)
   - Center: \((0, -5)\), Radius: 2
   - Graph:

3. \((x - 1)^2 + (y + 3)^2 = 9\)
   - Center: \((1, -3)\), Radius: 3
   - Graph:

4. \((x - 2)^2 + (y + 4)^2 = 16\)
   - Center: \((2, -4)\), Radius: 4
   - Graph:

5. \(x^2 + y^2 - 10x + 8y + 16 = 0\)
   - Center: \((5, -4)\), Radius: 5
   - Graph:

6. \(x^2 + y^2 - 4x + 6y = 12\)
   - Center: \((2, -3)\), Radius: 5
   - Graph:

**Skills Practice**

**Circles**

Write an equation for the circle that satisfies each set of conditions.

1. Center \((0, 5)\), radius 1 unit
   - \(x^2 + (y - 5)^2 = 1\)

2. Center \((5, 12)\), radius 8 units
   - \((x - 5)^2 + (y - 12)^2 = 64\)

3. Center \((4, 0)\), radius 2 units
   - \((x - 4)^2 + y^2 = 4\)

4. Center \((2, 2)\), radius 3 units
   - \((x - 2)^2 + (y - 2)^2 = 9\)

5. Center \((4, -4)\), radius 4 units
   - \((x - 4)^2 + (y + 4)^2 = 16\)

6. Center \((-6, 4)\), radius 5 units
   - \((x + 6)^2 + (y - 4)^2 = 25\)

7. Endpoints of a diameter at \((-12, 0)\) and \((12, 0)\)
   - \(x^2 + y^2 = 144\)

8. Endpoints of a diameter at \((-4, 0)\) and \((-4, -6)\)
   - \((x + 4)^2 + (y + 3)^2 = 9\)

9. Center at \((7, -3)\), passes through the origin \((x - 7)^2 + (y + 3)^2 = 58\)

10. Center at \((-4, 4)\), passes through \((-4, 1)\)
    - \((x + 4)^2 + (y - 4)^2 = 9\)

11. Center at \((-6, -5)\), tangent to \(y\)-axis \((x + 6)^2 + (y + 5)^2 = 36\)

12. Center at \((5, 1)\), tangent to \(x\)-axis \((x - 5)^2 + (y - 1)^2 = 1\)

Find the center and radius of the circle with the given equation. Then graph the circle.

13. \(x^2 + y^2 = 9\)
    - Center: \((0, 0)\), Radius: 3
    - Graph:

14. \((x - 1)^2 + (y - 2)^2 = 4\)
    - Center: \((1, 2)\), Radius: 2
    - Graph:

15. \((x + 1)^2 + y^2 = 16\)
    - Center: \((-1, 0)\), Radius: 4
    - Graph:

16. \(x^2 + (y + 3)^2 = 81\)
    - Center: \((0, -3)\), Radius: 9
    - Graph:

17. \((x - 5)^2 + (y + 8)^2 = 49\)
    - Center: \((5, -8)\), Radius: 7
    - Graph:

18. \(x^2 + y^2 - 4y - 32 = 0\)
    - Center: \((0, 2)\), Radius: 4
    - Graph:
Write an equation for the circle that satisfies each set of conditions.

1. center $(−4, 2)$, radius 8 units
   
   $$(x + 4)^2 + (y − 2)^2 = 64$$

2. center $(0, 0)$, radius 4 units
   
   $$x^2 + y^2 = 16$$

3. center $(−\frac{1}{2}, −\frac{\sqrt{3}}{2})$, radius $5\sqrt{2}$ units
   
   $$\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{\sqrt{3}}{2}\right)^2 = 50$$

4. center $(2.5, 4.2)$, radius 0.9 unit
   
   $$\left(x − 2.5\right)^2 + \left(y − 4.2\right)^2 = 0.81$$

Endpoints of a diameter at $(-2, −9)$ and $(0, −5)$

$$x^2 + y^2 + 7x + 6y = 25$$

Center at $(−6, 5)$, tangent to $x$-axis

$$x^2 + y^2 − 12x + 10y = 50$$

Find the center and radius of the circle with the given equation. Then graph the circle.

8. $(x + 3)^2 + y^2 = 16$

9. $3x^2 + y^2 = 12$

10. $x^2 + y^2 + 2x + 6y = 26$

11. $(x − 1)^2 + y^2 + 4y = 12$

12. $x^2 − 6x + y^2 = 0$

13. $x^2 + y^2 + 2x + 6y = −1$

### WEATHER

For Exercises 14 and 15, use the following information.

On average, the circular eye of a hurricane is about 15 miles in diameter. Gale winds can affect an area up to 300 miles from the storm’s center. In 2005, Hurricane Katrina devastated southern Louisiana. A satellite photo of Katrina’s landfall showed the center of its eye on one coordinate system could be approximated by the point $(80, 26)$.

14. Write an equation to represent a possible boundary of Katrina’s eye.

$$(x − 80)^2 + (y − 26)^2 = 56.25$$

15. Write an equation to represent a possible boundary of the area affected by gale winds.

$$(x − 80)^2 + (y − 26)^2 = 90,000$$

### POOLS

The pool on an architectural floor plan is given by the equation

$$x^2 + 6x + y^2 + 8y = 0$$

What point on the edge of the pool is farthest from the origin?

$$(−6, −8)$$

### TREASURE

For Exercises 5 and 6, use the following information.

A mathematically inclined pirate decided to hide the location of a treasure by marking it as the center of a circle given by an equation in non-standard form.

5. Rewrite the equation of the circle in standard form.

$$(x − 1)^2 + (y + 7)^2 = 1$$

6. Draw the circle on the map. Where is the treasure?

See circle on map at $(1, −7)$; the southwest corner of Meadow Madness.
Chapter 10

10-3 Enrichment

Tangents to Circles

A line that intersects a circle in exactly one point is a tangent to the circle. In the diagram, line $l$ is tangent to the circle with equation $x^2 + y^2 = 25$ at the point whose coordinates are $(3, 4)$.

A line is tangent to a circle at a point $P$ on the circle if and only if the line is perpendicular to the radius from the center of the circle to point $P$. This fact enables you to find an equation of the tangent to a circle at a point $P$ if you know an equation for the circle and the coordinates of $P$.

Exercises

1. What is the slope of the radius to the point with coordinates $(3, 4)$? What is the slope of the tangent to that point?

$\frac{4}{3} = \frac{-3}{4}$

2. Find an equation of the line $l$ that is tangent to the circle at $(3, 4)$.

$y = \frac{-3}{4}x + \frac{25}{4}$

3. If $k$ is a real number between -5 and 5, how many points on the circle have $x$-coordinate $k$? State the coordinates of these points in terms of $k$.

Two, $(k, \pm \sqrt{25 - k^2})$

4. Describe how you can find equations for the tangents to the points you named for Exercise 3.

Use the coordinates of $(0, 0)$ and of one of the given points. Find the slope of the radius to that point. Use the slope of the radius to find what the slope of the tangent must be. Use the slope of the tangent and the coordinates of the point on the circle to find an equation for the tangent.

5. Find an equation for the tangent at $(-3, 4)$.

$y = \frac{3}{4}x + \frac{25}{4}$

-5

Chapter 10

Graphing Calculator Activity

Matrices and Equations of Circles

A graphing calculator can be used to write the equation of a circle in the form $x^2 + y^2 + Dx + Ey + F = 0$ given any three points on the circle.

Example

Write the equation of the circle that passes through the given points. Identify the center and radius of each circle.

a. $A(5, 3), B(-2, 2),$ and $C(-1, -5)$

Substitute each ordered pair for $(x, y)$ in $x^2 + y^2 + Dx + Ey + F = 0$ to form the system of equations.

$5D + 3E + F = -34$

$-2D + 2E + F = -8$

$-D - 5E + F = -26$

Solve the system using a matrix equation to find $D, E, F$. Replace the coefficients in the expanded form. Then, complete the square to write the equation in standard form to identify the center and radius.

Keystrokes: $\text{[MATRX]} 3 \ 3 \ 5 \ \text{[EDIT]} 2 \ \text{[MATRX]}$ DO

$\text{[QUIT]}$ DO

Answers

(Exercise 10-3)

Lesson Reading Guide

Ellipses

Get Ready for the Lesson

Read the introduction to Lesson 10-4 in your textbook.

Is the Earth always the same distance from the Sun? Explain your answer using the words circle and ellipse. No; if the Earth’s orbit were a circle, it would always be the same distance from the Sun because every point on a circle is the same distance from the center. However, the Earth’s orbit is an ellipse, and the points on an ellipse are not all the same distance from the center.

Read the Lesson

1. An ellipse is the set of all points in a plane such that the sum of the distances from two fixed points is constant. The two fixed points are called the foci. Write an equation for the ellipse shown.

\[
\frac{x^2}{9} + \frac{y^2}{4} = 1
\]

2. Consider the ellipse with equation \(\frac{x^2}{9} + \frac{y^2}{4} = 1\).
   a. For this equation, \(a = 3\) and \(b = 2\).
   b. Write an equation that relates the values of \(a\), \(b\), and \(c\). \(c = a^2 - b^2\)
   c. Find the value of \(c\) for this ellipse. \(\sqrt{5}\)

3. Consider the ellipses with equations \(\frac{x^2}{25} + \frac{y^2}{16} = 1\) and \(\frac{x^2}{9} + \frac{y^2}{4} = 1\). Complete the following table to describe characteristics of their graphs.

<table>
<thead>
<tr>
<th>Standard Form of Equation</th>
<th>(\frac{x^2}{25} + \frac{y^2}{16} = 1)</th>
<th>(\frac{x^2}{9} + \frac{y^2}{4} = 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Direction of Major Axis</td>
<td>vertical</td>
<td>horizontal</td>
</tr>
<tr>
<td>Direction of Minor Axis</td>
<td>horizontal</td>
<td>vertical</td>
</tr>
<tr>
<td>Foci</td>
<td>((0, 3), (0, -3))</td>
<td>((-\sqrt{5}, 0), (\sqrt{5}, 0))</td>
</tr>
<tr>
<td>Length of Major Axis</td>
<td>10 units</td>
<td>6 units</td>
</tr>
<tr>
<td>Length of Minor Axis</td>
<td>8 units</td>
<td>4 units</td>
</tr>
</tbody>
</table>

Remember What You Learned

4. Some students have trouble remembering the two standard forms for the equation of an ellipse. How can you remember which term comes first and where to place \(a\) and \(b\) in these equations? The \(x\)-axis is horizontal. If the major axis is horizontal, the first term is \(\frac{x^2}{a^2}\). The \(y\)-axis is vertical. If the major axis is vertical, the first term is \(\frac{y^2}{b^2}\). \(a\) is always larger than \(b\).
Graph Ellipses

To graph an ellipse, if necessary, write the given equation in the standard form of an equation for an ellipse.

\[
\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \quad \text{(for ellipse with major axis horizontal)}
\]

\[
\frac{(y - k)^2}{b^2} + \frac{(x - h)^2}{a^2} = 1 \quad \text{(for ellipse with major axis vertical)}
\]

Use the center \((h, k)\) and the endpoints of the axes to plot four points of the ellipse. To make a more accurate graph, use a calculator to find some approximate values for \(x\) and \(y\) that satisfy the equation.

**Example**

Graph the ellipse \(4x^2 + 6y^2 + 8x - 36y = -34\).

\[
4x^2 + 6y^2 + 8x - 36y = -34
\]

4\((x + 1)^2 + 6(y - 3)^2 = 24
\]

The center of the ellipse is \((-1, 3)\). Since \(a^2 = 6, a = \sqrt{6}\).

The length of the major axis is \(2\sqrt{6}\), and the length of the minor axis is 4. Since the \(x\)-term has the greater denominator, the major axis is horizontal. Plot the endpoints of the axes. Then graph the ellipse.

**Exercises**

Find the coordinates of the center and the lengths of the major and minor axes for the ellipse with the given equation. Then graph the ellipse.

1. \(\frac{x^2}{12} + \frac{y^2}{9} = 1\) \((0, 0), 4\sqrt{3}, 6\)
2. \(\frac{x^2}{25} + \frac{y^2}{4} = 1\) \((0, 0), 10, 4\)
3. \(x^2 + 4y^2 + 24y = -32\) \((0, -3), 4, 2\)
4. \(9x^2 + 6y^2 - 36x + 12y = 12\) \((-1, 6), 2\sqrt{6}\)

Write an equation for each ellipse.

1. \(\frac{x^2}{9} + \frac{y^2}{4} = 1\)
2. \(\frac{y^2}{25} + \frac{x^2}{16} = 1\)
3. \(\frac{x^2}{16} + \frac{(y - 2)^2}{9} = 1\)

Write an equation for the ellipse that satisfies each set of conditions.

4. endpoints of major axis at \((0, 6)\) and \((0, -6)\), endpoints of minor axis at \((-3, 0)\) and \((3, 0)\)
5. endpoints of major axis at \((2, 6)\) and \((8, 6)\), endpoints of minor axis at \((5, 4)\) and \((5, 8)\)
6. endpoints of major axis at \((7, 0)\) and \((7, 9)\), endpoints of minor axis at \((5, 6)\) and \((9, 6)\)
7. major axis 12 units long and parallel to \(x\)-axis, minor axis 4 units long, center at \((0, 0)\)
8. endpoints of major axis at \((-6, 0)\) and \((6, 0)\), foci at \((-\sqrt{32}, 0)\) and \((\sqrt{32}, 0)\)
9. endpoints of major axis at \((0, 12)\) and \((0, -12)\), foci at \((0, \sqrt{32})\) and \((0, -\sqrt{32})\)

Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with the given equation. Then graph the ellipse.

10. \(\frac{x^2}{100} + \frac{y^2}{81} = 1\) \((0, 0); (0, \pm\sqrt{19}); 20, 18\)
11. \(\frac{x^2}{81} + \frac{y^2}{9} = 1\) \((0, 0); (\pm\sqrt{2}, 0); 18, 6\)
12. \(\frac{x^2}{49} + \frac{y^2}{25} = 1\) \((0, 0); (\pm\sqrt{4}, \pm\sqrt{5}); 14, 10\)
Chapter 10

10-4 Practice
Ellipses

Write an equation for each ellipse.

1. \( \frac{x^2}{121} + \frac{y^2}{9} = 1 \)
2. \( \frac{(y-2)^2}{9} + \frac{x^2}{4} = 1 \)
3. \( \frac{(x+1)^2}{25} + \frac{(y-3)^2}{9} = 1 \)

Write an equation for the ellipse that satisfies each set of conditions.

4. Endpoints of major axis at (-9, 0) and (9, 0), endpoints of minor axis at (0, 3) and (0, -3).
   \( \frac{x^2}{81} + \frac{y^2}{9} = 1 \)

5. Endpoints of major axis at (4, 2) and (4, -8), endpoints of minor axis at (1, -3) and (7, -3).
   \( \frac{(y+8)^2}{25} + \frac{(x-4)^2}{9} = 1 \)

6. Major axis 20 units long and parallel to x-axis, center at (2, -4).
   \( \frac{(y+4)^2}{25} + \frac{(x-2)^2}{9} = 1 \)

7. Major axis 10 units long, minor axis 6 units long and parallel to x-axis, center at (2, -4).
   \( \frac{(y+6)^2}{4} + \frac{(x-2)^2}{4} = 1 \)

8. Major axis 16 units long, center at (0, 0), foci at (0, 2\sqrt{15}) and (0, -2\sqrt{15}).
   \( \frac{x^2}{144} + \frac{y^2}{9} = 1 \)

9. Endpoints of minor axis at (0, 2) and (0, -2), foci at (-4, 0) and (4, 0).
   \( \frac{x^2}{100} + \frac{y^2}{25} = 1 \)

Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with the given equation. Then graph the ellipse.

10. \( \frac{x^2}{16} + \frac{y^2}{9} = 1 \)
    Center: (0, 0); Foci: (0, ±\sqrt{7}); Major axis: 4 units; Minor axis: 3 units

11. \( \frac{x^2}{36} + \frac{(y-3)^2}{1} = 1 \)
    Center: (3, 1); Foci: (3, 1 ± \sqrt{35}); Major axis: 6 units; Minor axis: 2 units

12. \( \frac{x^2}{49} + \frac{(y+3)^2}{25} = 1 \)
    Center: (-4, -3); Foci: (-4 ± 2\sqrt{6}, -3); Major axis: 10 units; Minor axis: 8 units

13. SPORTS An ice skater traces two congruent ellipses to form a figure eight. Assume that the center of the first loop is at the origin, with the second loop to its right. Write an equation to model the first loop if its major axis (along the x-axis) is 12 feet long and its minor axis is 6 feet long. Write another equation to model the second loop.
   \( \frac{x^2}{36} + \frac{y^2}{9} = 1; \frac{(x-12)^2}{36} + \frac{y^2}{9} = 1 \)

10-4 Word Problem Practice
Ellipses

1. PERSPECTIVE A graphic designer uses an ellipse to draw a circle from the horizontal perspective. The equation used is \( \frac{x^2}{25} + \frac{y^2}{9} = 1 \). Graph this ellipse.

2. ECHOES The walls of an elliptical room are given by the equation \( \frac{x^2}{25} + \frac{y^2}{16} = 1 \). Two people want to stand at the foci of the ellipse so that they can whisper to each other without anybody else hearing. What are the coordinates of the foci? (3, 0) and (-3, 0)

3. FLASHLIGHTS Daniella ended up doing her math homework late at night. To avoid disturbing others, she worked in bed with a pen light. One problem asked her to draw an ellipse. She noticed that her pen light created an elliptical patch of light on her paper, so she simply traced the outline of the patch of light.
   The outline of the ellipse is shown below. What is the equation of this ellipse in standard form?

4. ASTRONOMY The orbit of an asteroid is given by the equation \( \frac{x^2}{400} + \frac{y^2}{441} = 1 \), where each unit represents one astronomical unit (i.e. the distance from Sun to Earth). What are the lengths of the major and minor axes of the orbit?
   Major axis: 42 astronomical units, Minor axis: 40 astronomical units

MODELING For Exercises 5 and 6, use the following information.

James wants to try to make an ellipse using two tacks located at (5, 0) and (-5, 0), what is the equation of the ellipse in standard form?

5. Determine the lengths of the major and minor axes of the ellipse that James drew.
   Major axis has length 26, minor axis has length 24.

6. If a coordinate grid is overlaid on the ellipse so that the tacks are located at (5, 0) and (-5, 0), what is the equation of the ellipse in standard form?
   \( \frac{x^2}{169} + \frac{y^2}{144} = 1 \)
**Eccentricity**

In an ellipse, the ratio $\frac{c}{a}$ is called the **eccentricity** and is denoted by the letter $e$. Eccentricity measures the elongation of an ellipse. The closer $e$ is to 0, the more an ellipse looks like a circle. The closer $e$ is to 1, the more elongated it is. Recall that the equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a$ is the length of the major axis, and that $c = \sqrt{a^2 - b^2}$.

Find the eccentricity of each ellipse rounded to the nearest hundredth.

1. $\frac{x^2}{9} + \frac{y^2}{36} = 1$
   - $e = 0.87$

2. $\frac{x^2}{81} + \frac{y^2}{9} = 1$
   - $e = 0.94$

3. $\frac{x^2}{4} + \frac{y^2}{9} = 1$
   - $e = 0.75$

4. $\frac{x^2}{16} + \frac{y^2}{9} = 1$
   - $e = 0.66$

5. $\frac{x^2}{36} + \frac{y^2}{16} = 1$
   - $e = 0.75$

6. $\frac{x^2}{4} + \frac{y^2}{36} = 1$
   - $e = 0.94$

7. Is a circle an ellipse? Explain your reasoning.
   - Yes; it is an ellipse with eccentricity 0.

8. The center of the sun is one focus of Earth’s orbit around the sun. The length of the major axis is 186,000,000 miles, and the foci are 3,200,000 miles apart. Find the eccentricity of Earth’s orbit.
   - approximately 0.017

9. An artificial satellite orbiting the earth travels at an altitude that varies between 132 miles and 583 miles above the surface of the earth. If the center of the earth is one focus of its elliptical orbit and the radius of the earth is 3950 miles, what is the eccentricity of the orbit?
   - approximately 0.052

---

**Hyperbolas**

Get Ready for the Lesson

Read the introduction to Lesson 10-5 in your textbook.

Look at the sketch of a hyperbola in the introduction to this lesson. List three ways in which hyperbolas are different from parabolas.

Sample answer: A hyperbola has two branches, while a parabola is one continuous curve. A hyperbola has two foci, while a parabola has one focus. A hyperbola has two vertices, while a parabola has one vertex.

Read the Lesson

1. The graph at the right shows the hyperbola whose equation in standard form is $\frac{x^2}{16} - \frac{y^2}{9} = 1$.
   - The point (0, 0) is the **center** of the hyperbola.
   - The points (4, 0) and (-4, 0) are the **vertices** of the hyperbola.
   - The points (5, 0) and (-5, 0) are the **foci** of the hyperbola.
   - The segment connecting (4, 0) and (-4, 0) is called the **transverse** axis.
   - The segment connecting (0, 3) and (0, -3) is called the **conjugate** axis.
   - The lines $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$ are called the **asymptotes**.

2. Study the hyperbola graphed at the right.
   - The center is **(0, 0)**.
   - The value of $a$ is **2**.
   - The value of $c$ is **4**.
   - To find $b^2$, solve the equation $c^2 = a^2 + b^2$.
   - The equation in standard form for this hyperbola is $\frac{x^2}{4} - \frac{y^2}{12} = 1$.

Remember What You Learned

3. What is an easy way to remember the equation relating the values of $a$, $b$, and $c$ for a hyperbola? This equation looks just like the Pythagorean Theorem, although the variables represent different lengths in a hyperbola than in a right triangle.
Equations of Hyperbolas

A hyperbola is the set of all points in a plane such that the absolute value of the difference of the distances from any point on the hyperbola to any two given points in the plane, called the foci, is constant.

In the table, the lengths \(a, b,\) and \(c\) are related by the formula \(c^2 = a^2 + b^2.\)

<table>
<thead>
<tr>
<th>Standard Form of Equation</th>
<th>(\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1)</th>
<th>(\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equations of the Asymptotes</td>
<td>(y - k = \pm \frac{a}{b}(x - h))</td>
<td>(x - h = \pm \frac{b}{a}(y - k))</td>
</tr>
<tr>
<td>Vertices</td>
<td>((h - a, k), (h + a, k))</td>
<td>((h - b, k), (h + b, k))</td>
</tr>
<tr>
<td>Foci</td>
<td>((h - c, k), (h + c, k))</td>
<td>((h, k - c), (h, k + c))</td>
</tr>
</tbody>
</table>

**Example**

Write an equation for the hyperbola with vertices \((-2, 1)\) and \((6, 1)\) and foci \((-4, 1)\) and \((8, 1)\).

Use a sketch to orient the hyperbola correctly. The center of the hyperbola is the midpoint of the segment joining the two vertices. The center is \(\left(-\frac{2 + 6}{2}, \frac{1 + 1}{2}\right)\), or \((2, 1)\). The length of \(a\) is the distance from the center to a vertex, so \(a = 4\). The length of \(c\) is the distance from the center to a focus, so \(c = 6\).

\[
e^2 = a^2 + c^2 = 4^2 + 6^2 = 16 + 36 = 52
\]

Use \(h, k, a,\) and \(c\) to write an equation of the hyperbola.

\[
\frac{(x - 2)^2}{4^2} - \frac{(y - 1)^2}{6^2} = 1
\]

**Exercises**

Write an equation for the hyperbola that satisfies each set of conditions.

1. \((-7, 0)\) and \((7, 0)\), conjugate axis of length 10
2. \((-2, 3)\) and \((4, -3)\), foci \((-5, -3)\) and \((7, 3)\)
3. \((4, 3)\) and \((4, -5)\), conjugate axis of length 4
4. \((-8, 0)\) and \((8, 0)\), equation of asymptotes \(y = \pm \frac{1}{6}x\)
5. \((-4, 6)\) and \((-4, -2)\), foci \((-4, 10)\) and \((-4, -6)\)

Graph Hyperbolas

To graph a hyperbola, write the given equation in the standard form of an equation for a hyperbola.

\[
\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \quad \text{if the branches of the hyperbola open left and right, or}
\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1 \quad \text{if the branches of the hyperbola open up and down}
\]

Graph the point \((h, k)\), which is the center of the hyperbola. Draw a rectangle with dimensions \(2a\) and \(2b\) and center \((h, k)\).

**Example**

**Exercises**

Draw the graph of \(6y^2 - 4x^2 - 36y - 8x = -26\).

Complete the squares to get the equation in standard form.

\[
6(y^2 - 6y + \frac{9}{2}) - 4(x^2 + 2x) = -26 + \frac{81}{2} - 4\]

\[
\frac{(y - 3)^2}{\frac{25}{4}} - \frac{(x + 1)^2}{6} = 1
\]

The center of the hyperbola is \((-1, 3)\). According to the equation, \(a^2 = 4\) and \(b^2 = 6\), so \(a = 2\) and \(b = \sqrt{6}\).

The transverse axis is vertical, so the vertices are \((-1, 5)\) and \((-1, 1)\). Draw a rectangle with vertical dimension \(5\) and horizontal dimension \(\sqrt{6} = 4.9\). The diagonals of this rectangle are the asymptotes. The branches of the hyperbola open up and down. Use the vertices and the asymptotes to sketch the hyperbola.
Chapter 10

**Skills Practice**

### Hyperbolas

Write an equation for each hyperbola.

1. \[ \frac{x^2}{25} - \frac{y^2}{16} = 1 \]
2. \[ \frac{y^2}{36} - \frac{x^2}{25} = 1 \]
3. \[ \frac{x^2}{4} - \frac{y^2}{25} = 1 \]

Write an equation for the hyperbola that satisfies each set of conditions.

4. Vertices (-4, 0) and (4, 0), conjugate axis of length 8
   \[ \frac{x^2}{16} - \frac{y^2}{64} = 1 \]
5. Vertices (0, 6) and (0, -6), conjugate axis of length 14
   \[ \frac{y^2}{36} - \frac{x^2}{49} = 1 \]
6. Vertices (0, 3) and (0, -3), conjugate axis of length 10
   \[ \frac{y^2}{9} - \frac{x^2}{25} = 1 \]
7. Vertices (-2, 0) and (2, 0), conjugate axis of length 4
   \[ \frac{x^2}{4} - \frac{y^2}{1} = 1 \]
8. Vertices (-3, 0) and (3, 0), foci (±5, 0)
   \[ \frac{x^2}{9} - \frac{y^2}{25} = 1 \]
9. Vertices (0, 2) and (0, -2), foci (0, ±5)
   \[ \frac{x^2}{4} - \frac{y^2}{25} = 1 \]
10. Vertices (0, -2) and (6, -2), foci (3 ± \sqrt{13}, -2)
    \[ \frac{(x - 3)^2}{9} - \frac{(y + 2)^2}{4} = 1 \]

Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with the given equation. Then graph the hyperbola.

11. \[ \frac{x^2}{9} - \frac{y^2}{36} = 1 \]
    (-3, 0); (3, 0); (0, ±3); (0, ±6);
    \[ y = ±2x \]
12. \[ \frac{x^2}{49} - \frac{y^2}{9} = 1 \]
    (0, ±7); (0, ±1); (±4, 0); (±1, 0);
    \[ y = ±\frac{3}{7}x \]
13. \[ \frac{x^2}{16} - \frac{y^2}{4} = 1 \]
    (0, ±2); (±4, 0); (±1, 0);
    \[ y = ±\frac{1}{2}x \]

### Practice

Write an equation for each hyperbola.

1. \[ \frac{x^2}{9} - \frac{y^2}{16} = 1 \]
2. \[ \frac{(x - 2)^2}{9} - \frac{(x + 3)^2}{25} = 1 \]
3. \[ \frac{(x - 1)^2}{16} - \frac{(y + 2)^2}{4} = 1 \]

Write an equation for the hyperbola that satisfies each set of conditions.

4. Vertices (0, 7) and (0, -7), conjugate axis of length 18 units
   \[ \frac{y^2}{81} - \frac{x^2}{144} = 1 \]
5. Vertices (1, -1) and (1, 9), conjugate axis of length 6 units
   \[ \frac{(y + 1)^2}{9} - \frac{(x - 1)^2}{1} = 1 \]
6. Vertices (-5, 0) and (5, 0), foci (±√36, 0)
   \[ \frac{x^2}{25} - \frac{y^2}{1} = 1 \]
7. Vertices (1, 1) and (1, -3), foci (1, -1 ± √5)
   \[ \frac{(y + 1)^2}{4} - \frac{(x - 1)^2}{1} = 1 \]

Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with the given equation. Then graph the hyperbola.

8. \[ \frac{x^2}{4} - \frac{y^2}{4} = 1 \]
   (0, ±4); (0, ±2√5);
   \[ y = ±2x \]
9. \[ \frac{(y - 2)^2}{9} - \frac{(x - 1)^2}{4} = 1 \]
   (1, 3), (1, 1);
   \[ y = ±\frac{3}{2}x \]
10. \[ \frac{(y + 2)^2}{4} - \frac{(x - 3)^2}{4} = 1 \]
    (3, 0), (3, -4);
    \[ y = ±\frac{1}{2}(x - 1) \]
11. ASTRONOMY Astronomers use special X-ray telescopes to observe the sources of celestial X-rays. Some X-ray telescopes are fitted with a metal mirror in the shape of a hyperbola, which reflects the X-rays to a focus. Suppose the vertices of such a mirror are located at (-3, 0) and (3, 0), and one focus is located at (5, 0). Write an equation that models the hyperbola formed by the mirror.
   \[ \frac{x^2}{9} - \frac{y^2}{16} = 1 \]
Hyperbolas

1. LIGHTHOUSES The location of a lighthouse is represented by the origin of a coordinate plane. A boat in the distance appears to be on a collision course with the lighthouse. However, the boat veers off and turns away at the last moment, avoiding the rocky shallows. The path followed by the boat is modeled by a branch of the hyperbola with equation $\frac{x^2}{400} - \frac{y^2}{25} = 1$. If the unit length corresponds to a yard, how close did the boat come to the lighthouse? 30 yd

2. FIND THE ERROR Curtis was trying to write the equation for a hyperbola with a vertical transverse axis of length 10 and conjugate axis of length 6. The equation he got was $\frac{y^2}{25} - \frac{x^2}{9} = 1$. Did he make a mistake? If so, what did he do wrong?

3. MIRROR At a carnival, designers are planning a funhouse. They plan to put a large hyperbolic mirror inside this funhouse. They design the mirror’s hyperbolic cross section on graph paper using a hyperbola with a horizontal transverse axis. The asymptotes are to be $y = 9x$ and $y = -9x$ so the mirror is somewhat shallow. They also want the vertices to be 1 unit from the origin. What equation should they use for the hyperbola?

$$\frac{x^2}{1} - \frac{y^2}{9} = 1$$

4. ASTRONOMY Astronomers discover a new comet. They study its path and discover that it can be modeled by a branch of a hyperbola with equation $4x^2 - 40x - 25y^2 = 0$. Rewrite this equation in standard form and find the center of the hyperbola.

$$\frac{(x - 5)^2}{25} - \frac{y^2}{4} = 1, \text{ center at } (5, 0)$$

LIGHTNING For Exercises 5-7, use the following information.

Brittany and Kirk were talking on the phone when Brittany heard the thunder from a lightning bolt outside. Eight seconds later, she could hear the same thunder over the phone. Brittany and Kirk live 2 miles apart and sounds travels about 1 mile every 5 seconds.

5. On a coordinate plane, assume that Brittany is located at ($-1, 0$) and Kirk is located at ($1, 0$). Write an equation using the Distance Formula that describes the possible locations of the lightning strike.

$$\sqrt{(x - 1)^2 + y^2} - \sqrt{(x + 1)^2 + y^2} = \frac{8}{5}$$

6. Rewrite the equation you wrote for Exercise 5 so it is in the standard form for a hyperbola.

$$\frac{x^2}{\frac{4}{5}} - \frac{y^2}{\frac{3}{5}} = 1$$

7. Which branch of the hyperbola corresponds to the places where the lightning bolt might have struck? The left branch, the branch with negative x coordinates.

5. Make a conjecture about the asymptotes of rectangular hyperbolas.

The coordinate axes are the asymptotes.
10-6 Lesson Reading Guide

Conic Sections

Get Ready for the Lesson

Read the introduction to Lesson 10-6 in your textbook.
The figures in the introduction show how a plane can slice a double cone to form the conic sections. Name the conic section that is formed if the plane slices the double cone in each of the following ways:

- The plane is parallel to the base of the double cone and slices through one of the cones that form the double cone. circle
- The plane is perpendicular to the base of the double cone and slices through both of the cones that form the double cone. hyperbola

Read the Lesson

1. Name the conic section that is the graph of each of the following equations. Give the coordinate of the vertex if the conic section is a parabola and of the center if it is a circle, an ellipse, or a hyperbola.
   a. \( \frac{x^2 - 3y^2}{36} + \frac{(y + 5)^2}{15} = 1 \) ellipse; (3, -5)
   b. \( x = -2y + 1 \) circle; (7, 1)
   c. \( (x - 5)^2 - (y + 5)^2 = 1 \) hyperbola; (5, -5)
   d. \( (x + 6)^2 + (y - 2)^2 = 1 \) circle; (-6, 2)

2. Each of the following is the equation of a conic section. For each equation, identify the values of \( A \) and \( C \). Then, without writing the equation in standard form, state whether the graph of each equation is a parabola, circle, ellipse, or hyperbola.
   a. \( 2x^2 + y^2 - 6x + 8y + 12 = 0 \) \( A = \frac{2}{2}; C = \frac{1}{1} \); type of graph: ellipse
   b. \( 2x^2 + 3x - 2y - 5 = 0 \) \( A = \frac{2}{2}; C = \frac{0}{0} \); type of graph: parabola
   c. \( 5x^2 + 10x + 5y^2 - 20y + 1 = 0 \) \( A = \frac{5}{5}; C = \frac{5}{5} \); type of graph: circle
   d. \( x^2 - 2y^2 + 4x + 2y - 5 = 0 \) \( A = \frac{1}{1}; C = \frac{-1}{-1} \); type of graph: hyperbola

Remember What You Learned

3. What is an easy way to recognize that an equation represents a parabola rather than one of the other conic sections?

If the equation has an \( x^2 \) term and \( y \) term but no \( y^2 \) term, then the graph is a parabola. Likewise, if the equation has a \( y^2 \) term and \( x \) term but no \( x^2 \) term, then the graph is a parabola.
Conic Sections

Identify Conic Sections If you are given an equation of the form

\[ Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \]

you can determine the type of conic section just by considering the values of \( A \) and \( C \). Refer to the following chart.

<table>
<thead>
<tr>
<th>Relationship of ( A ) and ( C )</th>
<th>Type of Conic Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A = 0 ) or ( C = 0 ), but not both.</td>
<td>parabola</td>
</tr>
<tr>
<td>( A = C )</td>
<td>circle</td>
</tr>
<tr>
<td>( A ) and ( C ) have the same sign, but ( A + C )</td>
<td>ellipse</td>
</tr>
<tr>
<td>( A ) and ( C ) have opposite signs.</td>
<td>hyperbola</td>
</tr>
</tbody>
</table>

Example Without writing the equation in standard form, state whether the graph of each equation is a parabola, circle, ellipse, or hyperbola.

1. \( 3x^2 - 3y^2 + 5x + 12 = 0 \)
   - \( A = 3 \) and \( C = -3 \) have opposite signs, so the graph of the equation is a hyperbola.

2. \( y^2 = 7x - 2x + 13 \)
   - \( A = 0 \), so the graph of the equation is a parabola.

Exercises Without writing the equation in standard form, state whether the graph of each equation is a parabola, circle, ellipse, or hyperbola.

1. \( x^2 - 17x - 5y + 8 \)  
   - parabola
2. \( 2x^2 + 2y^2 - 3x + 4y - 5 \)  
   - circle
3. \( 4x^2 - 8x = 4y^2 - 6y + 10 \)  
   - hyperbola
4. \( 6y^2 - 18 = 24 - 4x^2 \)  
   - ellipse
5. \( x^2 + 4(y - y^2) + 2x - 1 \)  
   - ellipse
6. \( x = y^2 - 5y + x^2 - 5 \)  
   - circle
7. \( 3x^2 + 4y^2 = 50 + y^2 \)  
   - circle
8. \( 9x^2 - 9y = 3(3x - 3y^2) \)  
   - circle
9. \( 111 = 11x^2 + 10y^2 \)  
   - ellipse
10. \( 3x^2 - 4p^2 + 12 \)  
    - hyperbola

Chapter 10

10-6 Study Guide and Intervention (continued)

Chapter 10

10-6 Skills Practice

Write each equation in standard form. State whether the graph of the equation is a parabola, circle, ellipse, or hyperbola. Then graph the equation.

1. \( x^2 - 25y^2 = 25 \)  
   - hyperbola
2. \( 9x^2 + 4y^2 = 36 \)  
   - ellipse
3. \( x^2 + y^2 = 16 \)  
   - circle
4. \( x^2 + 8x + y^2 = 9 \)  
   - circle
5. \( x^2 + 2x - 15 = y \)  
   - parabola
6. \( 100x^2 + 25y^2 = 400 \)  
   - ellipse

Without writing the equation in standard form, state whether the graph of each equation is a parabola, circle, ellipse, or hyperbola.

7. \( 9x^2 + 4y^2 = 36 \)  
   - ellipse
8. \( x^2 + y^2 = 25 \)  
   - circle
9. \( y = x^2 + 2x \)  
   - parabola
10. \( y = 2x^2 - 4x - 4 \)  
    - parabola
11. \( 4y^2 - 25x^2 = 100 \)  
    - hyperbola
12. \( 16x^2 + y^2 = 16 \)  
    - ellipse
13. \( 16x^2 - 4y^2 = 64 \)  
    - hyperbola
14. \( 5x^2 + 5y^2 = 25 \)  
    - circle
15. \( 25x^2 + 9y^2 = 225 \)  
    - ellipse
16. \( 16x^2 = 4y^2 - 4x = 144 \)  
    - hyperbola
17. \( y = 4x^2 - 36x - 144 \)  
    - parabola
18. \( x^2 + y^2 + 2 = 0 \)  
    - circle
19. \( (x + 3)^2 + (y - 1)^2 = 4 \)  
    - circle
20. \( 25x^2 - 50y + 4x^2 = 75 \)  
    - ellipse
21. \( x^2 - 6y^2 + 9 = 0 \)  
    - hyperbola
22. \( x = y^2 + 5y - 6 \)  
    - parabola
23. \( (x + 5)^2 + y^2 = 10 \)  
    - circle
24. \( 25x^2 + 10y^2 - 250 = 0 \)  
    - ellipse
Chapter 10

10-6 Practice
Conic Sections

Write each equation in standard form. State whether the graph of the equation is a parabola, circle, ellipse, or hyperbola. Then graph the equation.

1. \( y^2 = -3x \) parabola
2. \( x^2 + y^2 + 6x = 7 \) circle
3. \( 5x^2 - 6y^2 - 30x - 12y = -9 \) hyperbola
4. \( 196x^2 = 1225 - 100x^2 \) ellipse

Without writing the equation in standard form, state whether the graph of each equation is a parabola, circle, ellipse, or hyperbola.

5. \( 3x^2 = 9 - 3y^2 - 6y \) circle
6. \( 9x^2 + y^2 + 54x - 6y = -81 \) ellipse
7. \( 6x^2 + 6y^2 = 36 \) circle
8. \( 4x^2 - y^2 = 16 \) hyperbola
9. \( 9x^2 + 16y^2 - 64y - 80 = 0 \) ellipse
10. \( 5x^2 + 5y^2 - 45 = 0 \) circle
11. \( x^2 + 2x = y \) parabola
12. \( 4x^2 - 36x^2 + 4x - 144 = 0 \) hyperbola
13. ASTRONOMY A satellite travels in an hyperbolic orbit. It reaches the vertex of its orbit at \((5, 0)\) and then travels along a path that gets closer and closer to the line \(y = -\frac{2}{5}x\).

Write an equation that describes the path of the satellite if the center of its hyperbolic orbit is at \((0, 0)\).

\( \frac{x^2}{25} - \frac{y^2}{4} = 1 \)

Chapter 10 46

10-6 Word Problem Practice
Conic Sections

1. MISSING INFORMATION Rick began reading a book on conic sections. He came to this passage and discovered an inkblot covering part of an equation.

For example, although it may not be obvious, the equation below describes a circle.

\[ 7x^2 - 12x - y^2 - 16y - 94 = 0 \]

To see that \(x^2 + y^2\) is a circle, observe that the equation is:

\[ \frac{(x - 2)^2}{4} + \frac{(y - 3)^2}{9} = 1 \]

Based on the information in the passage and your own knowledge of conic sections, what number is being covered by the inkblot?

2. HEADLIGHTS The light from the headlight of a car is in the shape of a cone. The axis of the cone is parallel to the ground. What shape does the edge of the lit region form on the road, assuming that the road is flat and level?

3. REASONING Jason has been struggling with conic sections. He decides he needs more practice, but he needs to have a way of making practice equations. He decides to use an equation of the form

\[ Ax^2 + By^2 = 1, \]

where \(A\) and \(B\) are determined by rolling a pair of dice. After several rolls, he begins to realize that this system is not good enough because some conic sections never appear. Which types of conic section cannot occur using his method?

parabolas and hyperbolas are not possible

4. MIRROR A painter used a can of spray paint to make an image. The boundary of the image is described by the equation

\[ 4x^2 - 16x + y^2 - 6y + 21 = 0. \]

Put this equation into standard form and describe whether the curve is a circle, ellipse, parabola, or hyperbola.

\[ (x - 2)^2 + \frac{(y - 3)^2}{4} = 1 \]

ellipse

NONSTANDARD EQUATIONS
For Exercises 5-7, use the following information.

Consider the equation \(xy = 1\).

5. Are there any solutions of this equation that lie on the \(x\)- or \(y\)-axis?

no

6. Sketch a graph of the solutions of the equation.

7. Assuming that the equation represents a conic section, based on the graph, which type of conic section is it?

a hyperbola
Parabolic Football

A parabola is defined as all the points \((x, y)\) in the plane whose distance from a fixed point, called the **focus**, is the same as its distance from a fixed line, called the **directrix**. Examples of parabolas are the cables on a suspension bridge, satellite dishes, and the flight path of a football during a kick-off.

At the kick-off at the beginning of a football game the ball is placed on the 40-yard line of the kicking team. Suppose the receiving team catches the ball on the goal line. Assume the 50-yard line has coordinates \((0, 0)\) and the 40-yard line of the kicking team has coordinates \((-10, 0)\).

1. Determine which equation of the parabola best describes this situation. For the other choices explain why they do not make sense to the situation.
   - a. \(y = 50(x + 10)\)
     Incorrect. This says the goal line is the 50-yard line
   - b. \(y = 50(x - 10)\)
     Incorrect. Wrong direction and way too high.
   - c. \(y = \frac{1}{50}(x - 50)\)
     Correct
   - d. \(y = -20(x - 100)\)
     Incorrect. Too high, not right yard lines.

2. During the same game, the quarterback throws a forward pass from the 50-yard line to his receiver on the 25-yard line. Assuming the ball follows the path of a parabola, write an equation modeling the flight path of the ball from quarterback to receiver.
   Answer may vary based on orientation.

3. The team did not pick up the first down, so they elect to try a field goal. Fortunately, one of the assistant coaches is a part-time mathematician and found an equation that describes their kicker as \(y = -0.2x^2 + 8x\). The line of scrimmage is the 25-yard line, so the ball will be placed on the 32-yard line for the kick. Add 10 more yards for the depth of the end zone (goal line to the goal post), making it a 42-yard field goal attempt. Will your kicker make it?
   Setting the vertex at the origin, then the focus is located at \((0, 1/8)\).
Systems of Quadratic Equations: Like systems of linear equations, systems of quadratic equations can be solved by substitution and elimination. If the graphs are a conic section and a line, the system will have 0, 1, or 2 solutions. If the graphs are two conic sections, the system will have 0, 1, 2, 3, or 4 solutions.

Example 1: Solve the system of equations:

\[ \begin{align*}
-2y + x &= 3 \\
\frac{x}{2} + \frac{y}{3} &= 2
\end{align*} \]

Rewrite the second equation as: \( \frac{x}{2} + \frac{y}{3} = 2 \). Add \( \frac{1}{2} \) to each side.

Substitute these values for \( \frac{x}{2} \): \( x = 4 \) or \( x = -3 \).

The solutions are \((4, -7)\) and \((-3, 0)\).

Example 2: Solve the system of inequalities by graphing:

\[ \begin{align*}
2x + 3y &\leq 6 \\
x - y &> 0
\end{align*} \]

The graph of \( 2x + 3y \leq 6 \) consists of all points on or inside the circle with center \((0, 0)\) and radius \( \frac{2}{\sqrt{13}} \). The graph of \( x - y > 0 \) consists of all points on or outside the circle with center \((0, 0)\) and radius \( \frac{2}{\sqrt{13}} \). The solution of the system is the set of points in both regions.

Example 3: Solve the system of inequalities by graphing:

\[ \begin{align*}
x^2 + y^2 &> 9 \\
x^2 - y^2 &< 1
\end{align*} \]

The graph of \( x^2 + y^2 > 9 \) consists of all points on or inside the circle with center \((0, 0)\) and radius \( \frac{3}{\sqrt{2}} \). The graph of \( x^2 - y^2 < 1 \) are the points “inside” but not on the branches of the hyperbola shown. The solution of the system is the set of points in both regions.

Exercises:

1. \( \begin{align*}
y &= x^2 - 5 \\
y &= x - 3
\end{align*} \)

(2, -1), (-1, -4)

2. \( \begin{align*}
x^2 + (y - 5)^2 &= 25 \\
y &= -x^2
\end{align*} \)

(0, 0)

3. \( \begin{align*}
x^2 + (y - 5)^2 &= 25 \\
y &= x^2
\end{align*} \)

(0, 0), (3, 9), (-3, 9)

4. \( \begin{align*}
x^2 + y^2 &= 9 \\
x^2 + y &= 3
\end{align*} \)

(0, 3), (\( \sqrt{5}, -2 \)), (\( -\sqrt{5}, -2 \))

5. \( \begin{align*}
x^2 - y^2 &= 1 \\
x^2 + y^2 &= 16
\end{align*} \)

\( \left( \frac{\sqrt{34}, \sqrt{30}}{2}, \frac{\sqrt{34}, -\sqrt{30}}{2} \right), \left( \frac{7 + \sqrt{29}, 1 + \sqrt{29}}{2}, \frac{7 - \sqrt{29}, 1 - \sqrt{29}}{2} \right) \)

6. \( \begin{align*}
y &= x - 3 \\
x^2 + y^2 &= 16
\end{align*} \)

\( \left( \frac{\sqrt{34}, \sqrt{30}}{2}, \frac{\sqrt{34}, -\sqrt{30}}{2} \right), \left( \frac{7 + \sqrt{29}, 1 + \sqrt{29}}{2}, \frac{7 - \sqrt{29}, 1 - \sqrt{29}}{2} \right) \)
**10-7 Skills Practice**

### Solving Quadratic Systems

Find the exact solution(s) of each system of equations.

1. $y = x - 2 \quad (0, -2), (1, -1) \quad 2. y = x + 3 \quad (-1, 2) \quad 3. y = 3x \quad (0, 0), \left(\frac{1}{9}, \frac{1}{3}\right)$
   
2. $y = x^2 - 2 \quad y = 2x^2 \quad (1.5, 4.5)$

4. $y = x (\sqrt{2}, \sqrt{2}) \quad 5. x = 5 \quad (-5, 0) \quad 6. y = 7 \quad \text{no solution}$

7. $y = 2x - 2 \quad y = -x + 1 \quad 8. x - y = 0 \quad (1, 2) \quad y^2 = 4x + 2$

9. $y = 2 - x \quad (0, 2), (3, -1) \quad y = x^2 - 4x + 2$

10. $y = 2x - 1 \quad \text{no solution} \quad 11. y = 3x^2 \quad (0, 0) \quad 12. y = x^2 + 1 \quad (-1, 2), \quad y = -x^2 + 3 \quad (1, 2)$

13. $y = -4x \quad (-1, -4), (1, 4) \quad 14. y = -1 \quad (0, -1) \quad 15. x^2 + y^2 = 36 \quad (-3, 0), \quad x^2 - 9y^2 = 9 \quad (3, 0)$

16. $3(x + 2)^2 - 4(x - 3)^2 = 12 \quad 17. x^2 - 4x^2 = 4 \quad (-2, 0), \quad y = -2x + 2 \quad (0, 2), (3, -4) \quad x^2 + y^2 = 4 \quad (2, 0)$

18. $y^2 - 4x^2 = 4 \quad \text{no solution} \quad 19. y = 2x$

Solve each system of inequalities by graphing.

19. $x \geq 3 - 2 \quad x^2 + y^2 < 16 \quad 20. y \leq x \quad y \geq 2x^2 + 4 \quad 21. y^2 + 9y^2 < 144 \quad x^2 + 8y^2 < 16$

---

**22. GARDENING** An elliptical garden bed has a path from point A to point B. If the bed can be modeled by the equation $x^2 + 3y^2 = 12$ and the path can be modeled by the line $y = -\frac{1}{3}x$, what are the coordinates of points A and B? (−3, 1) and (3, −1)
**10-7 Word Problem Practice**

**Solving Quadratic Systems**

1. **GRAPHIC DESIGN**
   A graphic designer is drawing an ellipse and a line. The ellipse is drawn so that it appears on top of the line. In order to determine if the line is partially covered by the ellipse, the program solves for simultaneous solutions of the equations of the line and the ellipse. Complete the following table.

<table>
<thead>
<tr>
<th>No. of Intersections</th>
<th>Covered? Y/N</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>N</td>
</tr>
<tr>
<td>1</td>
<td>N or Y ok</td>
</tr>
<tr>
<td>2</td>
<td>Y</td>
</tr>
</tbody>
</table>

2. **ORBITS**
   An asteroid travels in an elliptical orbit. If the orbit of Earth is also an ellipse, what is the maximum number of times the asteroid could cross the orbit of Earth? 4

3. **CIRCLES**
   An artist is commissioned to complete a painting of only circles. She wants to include all possible ways circles can relate. What are the possible numbers of intersection points between two circles? For each case, sketch two distinct circles that intersect with the corresponding number of points. Explain why more intersections are not possible.

   0 circles  
   1 circle  
   2 circles

   A circle is determined by 3 points. Therefore, any two circles that share 3 or more points must be identical and not distinct.

4. **COLLISION AVOIDANCE**
   An object is traveling along a hyperbola given by the equation \( \frac{x^2}{9} - \frac{y^2}{16} = 1 \). A probe is launched from the origin along a straight-line path. Mission planners want the probe to get closer and closer to the object, but never hit it. There are two straight lines that meet their criteria. What are they? \( y = 2x \) and \( y = -2x \)

**TANGENTS**

For Exercises 5 and 6, use the following information.

An architect wants a straight path to run from the origin of a coordinate plane to the edge of an elliptically shaped patio so that the pathway forms a tangent to the ellipse. The ellipse is given by the equation \( \frac{(x-6)^2}{12} + \frac{y^2}{96} = 1 \).

5. Using the equation \( y = mx \) to describe the path, substitute into the equation for the ellipse to get a quadratic equation in \( x \).

\[
(1 + \frac{m^2}{8})x^2 - 12x + 24 = 0
\]

6. Solve for \( m \) in the equation you found for Exercise 5.

\( m = 2 \) or \(-2\)

**Enrichment**

**Graphing Quadratic Equations with xy-Terms**

You can use a graphing calculator to examine graphs of quadratic equations that contain \( xy \)-terms.

**Example**

Use a graphing calculator to display the graph of \( x^2 + xy + y^2 = 4 \).

Solve the equation for \( y \) in terms of \( x \) by using the quadratic formula.

\[
y^2 + xy + (x^2 - 4) = 0
\]

To use the formula, let \( a = 1 \), \( b = x \), and \( c = (x^2 - 4) \).

\[
y = \frac{-x \pm \sqrt{x^2 - 4(1)(x^2 - 4)}}{2}
\]

To graph the equation on the graphing calculator, enter the two equations:

\[
y = \frac{-x + \sqrt{16 - 3x^2}}{2}
\]

\[
y = \frac{-x - \sqrt{16 - 3x^2}}{2}
\]

**Exercises**

Use a graphing calculator to graph each equation. State the type of curve each graph represents.

1. \( y^2 + xy = 8 \)  
   hyperbola

2. \( x^2 + y^2 - 2xy - x = 0 \)  
   parabola

3. \( x^2 - xy + y^2 = 15 \)  
   ellipse

4. \( x^2 + xy + y^2 = -9 \)  
   graph is \( \emptyset \)

5. \( 2x^2 - 2xy - y^2 + 4x = 20 \)  
   hyperbola

6. \( x^2 - xy - 2y^2 + 2x + 5y - 3 = 0 \)  
   two intersecting lines