OCR ADVANCED SUBSIDIARY GCE
IN MATHEMATICS (3890, 3891 and 3892)

OCR ADVANCED GCE
IN MATHEMATICS (7890, 7891 and 7892)

Specimen Question Papers and Mark Schemes

These specimen question papers and mark schemes are intended to accompany the OCR Advanced Subsidiary GCE and Advanced GCE specifications in Mathematics for teaching from September 2004.

Centres are permitted to copy material from this booklet for their own internal use.

The specimen assessment material accompanying the new specifications is provided to give centres a reasonable idea of the general shape and character of the planned question papers in advance of the first operational examination.
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INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer all the questions.
- Give non-exact numerical answers correct to 3 significant figures, unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are not permitted to use a calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.
- You are reminded of the need for clear presentation in your answers.
1. Write down the exact values of
   (i) $4^{-2}$, 
   (ii) $(2\sqrt{2})^2$, 
   (iii) $(1^3 + 2^3 + 3^3)^{\frac{1}{2}}$. 

2. (i) Express $x^2 - 8x + 3$ in the form $(x + a)^2 + b$.
   (ii) Hence write down the coordinates of the minimum point on the graph of $y = x^2 - 8x + 3$.

3. The quadratic equation $x^2 + kx + k = 0$ has no real roots for $x$.
   (i) Write down the discriminant of $x^2 + kx + k$ in terms of $k$.
   (ii) Hence find the set of values that $k$ can take.

4. Find $\frac{dy}{dx}$ in each of the following cases:
   (i) $y = 4x^3 - 1$,
   (ii) $y = x^2(x^2 + 2)$,
   (iii) $y = \sqrt{x}$

5. (i) Solve the simultaneous equations
   $y = x^2 - 3x + 2$, $y = 3x - 7$.
   (ii) What can you deduce from the solution to part (i) about the graphs of $y = x^2 - 3x + 2$ and $y = 3x - 7$?
   (iii) Hence, or otherwise, find the equation of the normal to the curve $y = x^2 - 3x + 2$ at the point (3, 2), giving your answer in the form $ax + by + c = 0$ where $a$, $b$ and $c$ are integers.
6  (i) Sketch the graph of \( y = \frac{1}{x} \), where \( x \neq 0 \), showing the parts of the graph corresponding to both positive and negative values of \( x \). [2]

(ii) Describe fully the geometrical transformation that transforms the curve \( y = \frac{1}{x} \) to the curve \( y = \frac{1}{x+2} \). Hence sketch the curve \( y = \frac{1}{x+2} \). [5]

(iii) Differentiate \( \frac{1}{x} \) with respect to \( x \). [2]

(iv) Use parts (ii) and (iii) to find the gradient of the curve \( y = \frac{1}{x+2} \) at the point where it crosses the \( y \)-axis. [3]

7

The diagram shows a circle which passes through the points \( A (2, 9) \) and \( B (10, 3) \). \( AB \) is a diameter of the circle.

(i) Calculate the radius of the circle and the coordinates of the centre. [4]

(ii) Show that the equation of the circle may be written in the form \( x^2 + y^2 - 12x - 12y + 47 = 0 \). [3]

(iii) The tangent to the circle at the point \( B \) cuts the \( x \)-axis at \( C \). Find the coordinates of \( C \). [6]
(i) Find the coordinates of the stationary points on the curve \( y = 2x^3 - 3x^2 - 12x - 7 \). [6]

(ii) Determine whether each stationary point is a maximum point or a minimum point. [3]

(iii) By expanding the right-hand side, show that
\[
2x^3 - 3x^2 - 12x - 7 = (x + 1)^2 (2x - 7).
\] [2]

(iv) Sketch the curve \( y = 2x^3 - 3x^2 - 12x - 7 \), marking the coordinates of the stationary points and the points where the curve meets the axes. [3]
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<th>Marks</th>
<th>Instructions</th>
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<tr>
<td>1</td>
<td>(i) (\frac{1}{16})</td>
<td>B1 1</td>
<td>For correct value (fraction or exact decimal)</td>
</tr>
<tr>
<td></td>
<td>(ii) 8</td>
<td>B1 1</td>
<td>For correct value 8 only</td>
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<tr>
<td></td>
<td>(iii) 6</td>
<td>M1 A1 2</td>
<td>For 1(^3) + 2(^3) + 3(^3) = 36 seen or implied</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>For correct value 6 only</td>
</tr>
<tr>
<td>2</td>
<td>(i) (x^2 - 8x + 3 = (x - 4)^2 - 13)</td>
<td>B1</td>
<td>For ((x - 4)^2) seen, or statement (a = -4)</td>
</tr>
<tr>
<td></td>
<td>i.e. (a = -4, b = -13)</td>
<td>M1</td>
<td>For use of (implied) relation (a^2 + b = 3)</td>
</tr>
<tr>
<td></td>
<td>(ii) Minimum point is ((4, -13))</td>
<td>B1 (^\wedge)</td>
<td>For (x)-coordinate equal to their ((-a))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B1 (^\wedge) 2</td>
<td>For (y)-coordinate equal to their (b)</td>
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<tr>
<td>3</td>
<td>(i) Discriminant is (k^2 - 4k)</td>
<td>M1 A1 2</td>
<td>For attempted use of the discriminant</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>For correct expression (in any form)</td>
</tr>
<tr>
<td></td>
<td>(ii) For no real roots, (k^2 - 4k &lt; 0)</td>
<td>M1</td>
<td>For stating their (\Delta &lt; 0)</td>
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<tr>
<td></td>
<td>Hence (k(k - 4) &lt; 0)</td>
<td>M1</td>
<td>For factorising attempt (or other soln method)</td>
</tr>
<tr>
<td></td>
<td>So (0 &lt; k &lt; 4)</td>
<td>A1</td>
<td>For both correct critical values 0 and 4 seen</td>
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<td></td>
<td></td>
<td>A1 4</td>
<td>For correct pair of inequalities</td>
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<tr>
<td>4</td>
<td>(i) (\frac{dy}{dx} = 12x^2)</td>
<td>M1</td>
<td>For clear attempt at (nx^{n-1})</td>
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<tr>
<td></td>
<td></td>
<td>A1 2</td>
<td>For completely correct answer</td>
</tr>
<tr>
<td></td>
<td>(ii) (y = x^4 + 2x^2)</td>
<td>B1</td>
<td>For correct expansion</td>
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<td></td>
<td>Hence (\frac{dy}{dx} = 4x^3 + 4x)</td>
<td>M1</td>
<td>For correct differentiation of at least one term</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A1 (^\wedge) 3</td>
<td>For correct differentiation of their 2 terms</td>
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<tr>
<td></td>
<td>(iii) (\frac{dy}{dx} = \frac{1}{2}x^{-\frac{3}{2}})</td>
<td>M1</td>
<td>For clear differentiation attempt of (x^{\frac{1}{2}})</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A1 2</td>
<td>For correct answer, in any form</td>
</tr>
<tr>
<td>5</td>
<td>(i) (x^2 - 3x + 2 = 3x - 7 \Rightarrow x^2 - 6x + 9 = 0)</td>
<td>M1</td>
<td>For equating two expressions for (y)</td>
</tr>
<tr>
<td></td>
<td>Hence ((x - 3)^2 = 0)</td>
<td>A1</td>
<td>For correct 3-term quadratic in (x)</td>
</tr>
<tr>
<td></td>
<td>So (x = 3) and (y = 2)</td>
<td>M1</td>
<td>For factorising, or other solution method</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A1</td>
<td>For correct value of (x)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A1 5</td>
<td>For correct value of (y)</td>
</tr>
<tr>
<td></td>
<td>(ii) The line (y = 3x - 7) is the tangent to the curve (y = x^2 - 3x + 2) at the point ((3, 2))</td>
<td>B1</td>
<td>For stating tangency</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B1 2</td>
<td>For identifying (x = 3, y = 2) as coordinates</td>
</tr>
<tr>
<td></td>
<td>(iii) Gradient of tangent is 3</td>
<td>B1</td>
<td>For stating correct gradient of given line</td>
</tr>
<tr>
<td></td>
<td>Hence gradient of normal is (-\frac{4}{3})</td>
<td>B1 (^\wedge)</td>
<td>For stating corresponding perpendicular grad</td>
</tr>
<tr>
<td></td>
<td>Equation of normal is (y - 2 = -\frac{4}{3}(x - 3))</td>
<td>M1</td>
<td>For appropriate use of straight line equation</td>
</tr>
<tr>
<td></td>
<td>i.e. (x + 3y - 9 = 0)</td>
<td>A1 4</td>
<td>For correct equation in required form</td>
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6 (i) Translation of 2 units in the negative $x$-direction

(ii) Derivative is $-x^2$

(iii) Gradient of $y = \frac{1}{x}$ at $x = 2$ is required

(iv) Gradient of $AB$ is $\frac{3}{4}$

7 (i) $AB^2 = (10 - 2)^2 + (3 - 9)^2 = 100$

(ii) Equation is $(x - 6)^2 + (y - 6)^2 = 5^2$

(iii) Gradient of $AB$ is $\frac{3}{4}$

Hence perpendicular gradient is $\frac{4}{3}$

Equation of tangent is $y - 3 = \frac{4}{3}(x - 10)$

Hence $C$ is the point $(\frac{31}{4}, 0)$

For correct 1st quadrant branch

For both branches correct and nothing else

For translation parallel to the $x$-axis

For correct magnitude

For correct direction

For correct sketch of new curve

For some indication of location, e.g. $\frac{1}{2}$ at $y$-intersection or $-2$ at asymptote

For correct power $-2$ in answer

For correct coefficient $-1$

For correctly using the translation

For substituting $x = 2$ in their (iii)

For correct answer

For correct calculation method for $AB^2$

For correct value for radius

For correct calculation method for mid-point

For both coordinates correct

For using correct basic form of circle eqn

For expanding at least one bracket correctly

For showing given answer correctly

For finding the gradient of $AB$

For correct value $-\frac{3}{4}$ or equivalent

For relevant perpendicular gradient

For using their perp grad and $B$ correctly

For substituting $y = 0$ in their tangent eqn

For correct value $x = \frac{31}{4}$
### Question 8

#### (i)

\[
\frac{dy}{dx} = 6x^2 - 6x - 12
\]

Hence \( x^2 - x - 2 = 0 \)

\((x - 2)(x + 1) = 0 \Rightarrow x = 2 \text{ or } -1\)

Stationary points are \((2, -27)\) and \((-1, 0)\)

- **M1** For differentiation with at least 1 term OK
- **A1** For completely correct derivative
- **M1** For equating their derivative to zero
- **M1** For factorising or other solution method
- **A1** For both correct \(x\)-coordinates

#### (ii)

\[
\frac{d^2y}{dx^2} = 12x - 6 = \begin{cases} 
+18 \text{ when } x = 2 \\
-18 \text{ when } x = -1
\end{cases}
\]

Hence \((2, -27)\) is a min and \((-1, 0)\) is a max

- **M1** For attempt at second derivative and at least one relevant evaluation
- **A1** For either one correctly identified
- **A1** For both correctly identified

#### (iii)

\[
\text{RHS} = (x^2 + 2x + 1)(2x - 7)
\]

\[
= 2x^3 - 7x^2 + 4x^2 - 14x + 2x - 7
\]

\[
= 2x^3 - 3x^2 - 12x - 7, \text{ as required}
\]

- **M1** For squaring correctly and attempting complete expansion process
- **A1** For obtaining given answer correctly

#### (iv)

- **B1** For correct cubic shape
- **B1** For maximum point lying on \(x\)-axis
- **B1** For \(x = \frac{7}{2}\) and \(y = -7\) at intersections
OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MATHEMATICS

Core Mathematics 2

Specimen Paper

Additional materials:
Answer booklet
Graph paper
List of Formulae (MF 1)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

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• You are permitted to use a graphic calculator in this paper.

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• The total number of marks for this paper is 72.
• Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.
• You are reminded of the need for clear presentation in your answers.
1 Expand \((1 - 2x)^4\) in ascending powers of \(x\), simplifying the coefficients. [5]

2 (i) Find \(\int \frac{1}{x^2} \, dx\). [3]

(ii) The gradient of a curve is given by \(\frac{dy}{dx} = \frac{1}{x^2}\). Find the equation of the curve, given that it passes through the point \((1, 3)\). [3]

3 (a) Express each of the following in terms of \(\log_2 x\):

   (i) \(\log_2(x^2)\), [1]

   (ii) \(\log_2(8x^2)\). [3]

(b) Given that \(y^2 = 27\), find the value of \(\log_3 y\). [3]

4 Records are kept of the number of copies of a certain book that are sold each week. In the first week after publication 3000 copies were sold, and in the second week 2400 copies were sold. The publisher forecasts future sales by assuming that the number of copies sold each week will form a geometric progression with first two terms 3000 and 2400. Calculate the publisher’s forecasts for

   (i) the number of copies that will be sold in the 20th week after publication, [3]

   (ii) the total number of copies sold during the first 20 weeks after publication, [2]

   (iii) the total number of copies that will ever be sold. [2]

5 (i) Show that the equation \(15\cos^2\theta + 13 + \sin\theta = 0\) may be written as a quadratic equation in \(\sin\theta\). [2]

(ii) Hence solve the equation, giving all values of \(\theta\) such that \(0 \leq \theta \leq 360\). [6]
6

The diagram shows triangle $ABC$, in which $AB = 3$ cm, $AC = 5$ cm and angle $ABC = 2.1$ radians. Calculate

(i) angle $ACB$, giving your answer in radians, [2]

(ii) the area of the triangle. [3]

An arc of a circle with centre $A$ and radius 3 cm is drawn, cutting $AC$ at the point $D$.

(iii) Calculate the perimeter and the area of the sector $ABD$. [4]

7

The diagram shows the curves $y = -3x^2 - 9x + 30$ and $y = x^2 + 3x - 10$.

(i) Verify that the curves intersect at the points $A(-5, 0)$ and $B(2, 0)$. [2]

(ii) Show that the area of the shaded region between the curves is given by $\int_{-5}^{2} (-4x^2 - 12x + 40) \, dx$. [2]

(iii) Hence or otherwise show that the area of the shaded region between the curves is $228 \frac{2}{3}$. [5]
The diagram shows the curve \( y = 1.25^x \).

(i) A point on the curve has \( y \)-coordinate 2. Calculate its \( x \)-coordinate. [3]

(ii) Use the trapezium rule with 4 intervals to estimate the area of the shaded region, bounded by the curve, the axes, and the line \( x = 4 \). [4]

(iii) State, with a reason, whether the estimate found in part (ii) is an overestimate or an underestimate. [2]

(iv) Explain briefly how the trapezium rule could be used to find a more accurate estimate of the area of the shaded region. [1]

The cubic polynomial \( x^3 + ax^2 + bx - 6 \) is denoted by \( f(x) \).

(i) The remainder when \( f(x) \) is divided by \((x - 2)\) is equal to the remainder when \( f(x) \) is divided by \((x + 2)\). Show that \( b = -4 \). [3]

(ii) Given also that \((x - l)\) is a factor of \( f(x) \), find the value of \( a \). [2]

(iii) With these values of \( a \) and \( b \), express \( f(x) \) as a product of a linear factor and a quadratic factor. [3]

(iv) Hence determine the number of real roots of the equation \( f(x) = 0 \), explaining your reasoning. [3]
1. \( 1 - 8x + 24x^2 - 32x^3 + 16x^4 \)  
   B1 \( x \) For first two terms 1 – 8x  
   M1 \( x \) For expansion in powers of \((-2x)\)  
   M1 \( x \) For any correct use of binomial coefficients  
   A1 \( x \) For any one further term correct  
   A1 \( x \) For completely correct expansion  

2. (i) \( \int x^{-2} \, dx = -x^{-1} + c \)  
   B1 \( x \) For any attempt to integrate \( x^{-2} \)  
   A1 \( x \) For correct expression \(-x^{-1}\) (in any form)  
   B1 \( x \) For adding an arbitrary constant  
   
   (ii) \( y = -x^{-1} + c \) passes through \((1, 3)\),  
   so \( 3 = -1 + c \Rightarrow c = 4 \)  
   B1 \( x \) For attempt to use \((1, 3)\) to evaluate \(c\)  
   A1 \( x \) For correct value from their equation  
   
   Hence curve is \( y = -\frac{1}{x} + 4 \)  
   A1 \( x \) For correct equation  

3. (a) (i) \( 2 \log_2 x \)  
   B1 \( x \) For correct answer  
   
   (ii) \( \log_2(8x^2) = \log_2 8 + \log_2 x^2 = 3 + 2 \log_2 x \)  
   M1 \( x \) For relevant sum of logarithms  
   M1 \( x \) For relevant use of \(8 = 2^3\)  
   A1 \( x \) For correct simplified answer  
   
   (b) \( 2 \log_3 y = \log_3 27 \)  
   Hence \( \log_3 y = \frac{3}{2} \)  
   A1 \( x \) For any correct expression for \(\log_3 y\)  
   A1 \( x \) For correct simplified answer  

4. (i) \( r = \frac{2400}{3000} = 0.8 \)  
   B1 \( x \) For the correct value of \(r\)  
   
   Forecast for week 20 is \(3000 \times 0.8^{10} = 43\)  
   M1 \( x \) For correct use of \(ar^{n-1}\)  
   A1 \( x \) For correct (integer) answer  
   
   (ii) \( \frac{3000(1 - 0.8^{10})}{1 - 0.8} = 14827 \)  
   M1 \( x \) For correct use of \(\frac{a(1 - r^n)}{1 - r}\)  
   A1 \( x \) For correct answer (3sf is acceptable)  
   
   (iii) \( \frac{3000}{1 - 0.8} = 15000 \)  
   M1 \( x \) For correct use of \(\frac{a}{1 - r}\)  
   A1 \( x \) For correct answer  

5. (i) \( LHS = 15(1 - \sin^2 \theta) \)  
   Hence equation is \( 15 \sin^2 \theta + \sin \theta - 2 = 0 \)  
   A1 \( x \) For using the relevant trig identity  
   A1 \( x \) For correct 3-term quadratic  
   
   (ii) \( (5 \sin \theta + 2)(3 \sin \theta - 1) = 0 \)  
   Hence \( \sin \theta = -\frac{2}{5} \) or \( \frac{1}{3} \)  
   A1 \( x \) For both correct values  
   
   So \( \theta = 19.5, 160.5, 203.6, 336.4 \)  
   M1 \( x \) For any relevant inverse sine operation  
   A1 \( x \) For any one correct value  
   A1 \( x \) For corresponding second value  
   A1 \( x \) For both remaining values
6 (i) \( \frac{3}{\sin C} = \frac{5}{\sin 2.1} \Rightarrow \sin C = \frac{3}{5} \sin 2.1 \)

Hence \( C = 0.544 \) 

(ii) Angle \( A \) is \( \pi - 2.1 - 0.544 = 0.4972 \)

Area is \( \frac{1}{2} \times 5 \times 3 \times \sin 0.4972 \)

i.e. \( 3.58 \text{ cm}^2 \)

(iii) Sector perimeter is \( 6 + 3 \times 0.4972 \)

i.e. 7.49 cm

Sector area is \( \frac{1}{2} \times 3^2 \times 0.4972 \)

i.e. 2.24 cm²

7 (i) \(-75 + 45 + 30 = 0, 25 - 15 - 10 = 0\)

\(-12 - 18 + 30 = 0, 4 + 6 - 10 = 0\)

\[-12 = 0, 10 = 0\]

(ii) Area is \( \int_{-\infty}^{\infty} \left((-3x^2 - 9x + 30) - (x^2 + 3x - 10)\right) \, dx \)

i.e. \( \int_{-\infty}^{\infty} (-4x^2 - 12x + 40) \, dx \), as required

(iii) EITHER: Area is \( \left[-\frac{3}{4}x^3 - 6x^2 + 40x\right]_0^5 \)

\( = (-\frac{3}{4} \cdot 5^3 - 24 + 80) - (-\frac{3}{4} \cdot 0 - 150 - 200) \)

\( = 228 \frac{4}{3} \)

OR: Area under top curve is

\( \left[-x^3 - \frac{2}{3}x^2 + 30x\right]_0^5 = 171 \frac{1}{2} \)

Area above lower curve is

\( \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 - 10x\right]_0^5 = 57 \frac{1}{6} \)

So area between is \( 171 \frac{1}{2} + 57 \frac{1}{6} = 228 \frac{4}{3} \)

8 (i) \( 1.25^x = 2 \Rightarrow x \log 1.25 = \log 2 \)

Hence \( x = \frac{\log 2}{\log 1.25} = 3.11 \)

(ii) \( \frac{1}{2}(1.25^0 + 2(1.25^1 + 1.25^2 + 1.25^3)) + 1.25^4 \)

Area is 6.49

(iii) The trapezia used in (ii) extend above the curve

Hence the trapezium rule overestimates the area

(iv) Use more trapezia, with a smaller value of \( h \)
|   | \[8 + 4a + 2b - 6 = -8 + 4a - 2b - 6\] | M1 | For equating \(f(2)\) and \(f(-2)\)  
A1 | For correct equation  
A1 | For showing given answer correctly  
<table>
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<td>(4b = -16 \Rightarrow b = -4)</td>
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| (ii) | \[1 + a - 4 - 6 = 0\] | M1 | For equating \(f(1)\) to 0 (not \(f(-1)\))  
A1 | For correct value  
|   | \(Hence\ a = 9\) |   |   |   |   |
| (iii) | \(f(x) = (x - 1)(x^2 + 10x + 6)\) | M1 | For quadratic factor with \(x^2\) and/or +6 OK  
A1 | For trinomial with both these terms correct  
A1 | For completely correct factorisation  
| (iv) | The discriminant of the quadratic is 76  
Hence there are 3 real roots altogether | M1 | For evaluating the discriminant  
M1 | For using positive discriminant to deduce that there are 2 roots from the quadratic factor  
A1 | For completely correct explanation of 3 roots  

**Total**: 11 marks
OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MATHEMATICS 4723

Core Mathematics 3

Specimen Paper

Additional materials:
Answer booklet
Graph paper
List of Formulae (MF 1)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

• Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
• Answer all the questions.
• Give non-exact numerical answers correct to 3 significant figures, unless a different degree of accuracy is specified in the question or is clearly appropriate.
• You are permitted to use a graphic calculator in this paper.

INFORMATION FOR CANDIDATES

• The number of marks is given in brackets [ ] at the end of each question or part question.
• The total number of marks for this paper is 72.
• Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.
• You are reminded of the need for clear presentation in your answers.
1 Solve the inequality $|2x + 1| > |x - 1|$.

2 (i) Prove the identity

$$\sin(x + 30^\circ) + (\sqrt{3})\cos(x + 30^\circ) \equiv 2\cos x,$$

where $x$ is measured in degrees. 

(ii) Hence express $\cos 15^\circ$ in surd form.

3 The sequence defined by the iterative formula

$$x_{n+1} = \sqrt[4]{17 - 5x_n},$$

with $x_1 = 2$, converges to $\alpha$.

(i) Use the iterative formula to find $\alpha$ correct to 2 decimal places. You should show the result of each iteration.

(ii) Find a cubic equation of the form

$$x^3 + cx + d = 0$$

which has $\alpha$ as a root.

(iii) Does this cubic equation have any other real roots? Justify your answer.

4 The diagram shows the curve

$$y = \frac{1}{\sqrt{(4x+1)}}.$$

The region $R$ (shaded in the diagram) is enclosed by the curve, the axes and the line $x = 2$.

(i) Show that the exact area of $R$ is 1.

(ii) The region $R$ is rotated completely about the $x$-axis. Find the exact volume of the solid formed.
At time $t$ minutes after an oven is switched on, its temperature $\theta ^\circ C$ is given by 

$$\theta = 200 - 180e^{-0.1t}.$$ 

(i) State the value which the oven’s temperature approaches after a long time. [1] 

(ii) Find the time taken for the oven’s temperature to reach $150^\circ C$. [3] 

(iii) Find the rate at which the temperature is increasing at the instant when the temperature reaches $150^\circ C$. [4] 

The function $f$ is defined by 

$$f : x \mapsto 1 + \sqrt{x}, \quad \text{for } x \geq 0.$$ 

(i) State the domain and range of the inverse function $f^{-1}$. [2] 

(ii) Find an expression for $f^{-1}(x)$. [2] 

(iii) By considering the graphs of $y = f(x)$ and $y = f^{-1}(x)$, show that the solution to the equation 

$$f(x) = f^{-1}(x)$$ 

is $x = \frac{1}{4}(3 + \sqrt{5})$. [4] 

(i) Write down the formula for $\tan 2x$ in terms of $\tan x$. [1] 

(ii) By letting $\tan x = t$, show that the equation 

$$4 \tan 2x + 3 \cot x \sec^2 x = 0$$ 

becomes 

$$3t^4 - 8t^2 - 3 = 0.$$ [4] 

(iii) Hence find all the solutions of the equation 

$$4 \tan 2x + 3 \cot x \sec^2 x = 0$$ 

which lie in the interval $0 \leq x \leq 2\pi$. [4]
The diagram shows the curve \( y = (\ln x)^2 \).

(i) Find \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \). \[4\]

(ii) The point \( P \) on the curve is the point at which the gradient takes its maximum value. Show that the tangent at \( P \) passes through the point \((0, -1)\). \[6\]

The diagram shows the curve \( y = \tan^{-1} x \) and its asymptotes \( y = \pm a \).

(i) State the exact value of \( a \). \[1\]

(ii) Find the value of \( x \) for which \( \tan^{-1} x = \frac{1}{2} a \). \[2\]

The equation of another curve is \( y = 2\tan^{-1}(x-1) \).

(iii) Sketch this curve on a copy of the diagram, and state the equations of its asymptotes in terms of \( a \). \[3\]

(iv) Verify by calculation that the value of \( x \) at the point of intersection of the two curves is 1.54, correct to 2 decimal places. \[2\]

Another curve (which you are not asked to sketch) has equation \( y = \left(\tan^{-1} x\right)^2 \).

(v) Use Simpson’s rule, with 4 strips, to find an approximate value for \( \int_0^1 \left(\tan^{-1} x\right)^2 \, dx \). \[3\]
### Specimen Paper

1. **EITHER:**
   \[
   4x^2 + 4x + 1 > x^2 - 2x + 1
   \]
   i.e. \(3x^2 + 6x > 0\)
   So \(x(x + 2) > 0\)
   Hence \(x < -2\) or \(x > 0\)
   
   **OR:**
   Critical values where \(2x + 1 = \pm (x - 1)\)
   i.e. where \(x = -2\) and \(x = 0\)
   Hence \(x < -2\) or \(x > 0\)
   
   For squaring both sides
   For reduction to correct quadratic
   For factorising, or equivalent
   For both critical values correct
   For completely correct solution set

2. (i) \(\sin \left(\frac{x}{2}\right) + \cos \left(\frac{x}{2}\right) + (\sqrt{3})\cos \left(\frac{x}{2}\right)\sin \left(\frac{x}{2}\right) - \sin \left(\frac{x}{2}\right) = \frac{1}{2}\cos x + \frac{3}{2}\cos x = 2\cos x\)
   as required
   For expanding both compound angles
   For completely correct expansion
   For using exact values of \(\sin 30^\circ\) and \(\cos 30^\circ\)
   For showing given answer correctly

   (ii) \(\sin 45^\circ + (\sqrt{3})\cos 45^\circ = 2\cos 15^\circ\)
   Hence \(\cos 15^\circ = \frac{1 + \sqrt{3}}{2\sqrt{2}}\)
   For letting \(x = 15^\circ\) throughout
   For any correct exact form

3. (i) \(x_3 = \sqrt[3]{7} = 1.9129...\)
   \(x_3 = 1.9517...\)
   \(x_4 = 1.9346...\)
   \(\alpha = 1.94\) to 2dp

   (ii) \(x = \sqrt[3]{17 - 5x} \Rightarrow x^3 + 5x - 17 = 0\)
   For letting \(x_0 = x_{n+1} = x\) (or \(\alpha\))
   For correct equation stated

   (iii) **EITHER:** Graphs of \(y = x^3\) and \(y = 17 - 5x\) only cross once
   Hence there is only one real root
   For argument based on sketching a pair of graphs, or a sketch of the cubic by calculator
   For correct conclusion for a valid reason

   **OR:**
   \[\frac{d}{dx}(x^3 + 5x - 17) = 3x^2 + 5 > 0\]
   Hence there is only one real root
   For consideration of the cubic’s gradient
   For correct conclusion for a valid reason

4. (i) \(\int_0^1 (4x + 1)^{\frac{1}{2}} dx = \left[\frac{1}{2}(4x + 1)^{\frac{1}{2}}\right]_0^1 = \frac{1}{2}(3 - 1) = 1\)
   For integral of the form \(k(4x + 1)^{\frac{1}{2}}\)
   For correct indefinite integral
   For correct use of limits
   For given answer correctly shown

   (ii) \(\pi \int_0^1 \frac{1}{4x + 1} dx = \pi \left[\frac{1}{4}\ln(4x + 1)\right]_0^1 = \frac{1}{4}\pi \ln 9\)
   For integral of the form \(k \ln(4x + 1)\)
   For correct \(\frac{1}{4}\ln(4x + 1)\), with or without \(\pi\)
   Correct use of limits and \(\pi\)
   For correct (simplified) exact value
### Question 5

**(i)** 
\[ 200^\circ C \]

B1 1 For value 200

**(ii)** 
\[ 150 = 200 - 180e^{-0.1t} \Rightarrow e^{-0.1t} = \frac{50}{180} \]

Hence \(-0.1t = \ln \frac{5}{18} \Rightarrow t = 12.8 \)

M1 For isolating the exponential term
M1 For taking logs correctly
A1 3 For correct value 12.8 (minutes)

**(iii)** 
\[ \frac{d\theta}{dt} = 18e^{-0.1t} \]

Hence rate is \(18e^{-0.1\times12.8} = 5.0^\circ C \) per minute

M1 For differentiation attempt
A1 For correct derivative
M1 For using their value from (ii) in their \( \dot{\theta} \)
A1 4 For value 5.0(0)

### Question 6

**(i)**

Domain of \( f^{-1} \) is \( x \geq 1 \)

B1 For the correct set, in any notation

Range is \( x \geq 0 \)

B1 Ditto

**(ii)** 
If \( y = 1 + \sqrt{x} \), then \( x = (y - 1)^2 \)

Hence \( f^{-1}(x) = (x - 1)^2 \)

M1 For changing the subject, or equivalent
A1 2 For correct expression in terms of \( x \)

**(iii)**
The graphs intersect on the line \( y = x \)

Hence \( x \) satisfies \( x = (x - 1)^2 \)

\( i.e. \) \( x^2 - 3x + 1 = 0 \) \( \Rightarrow x = \frac{3 \pm \sqrt{5}}{2} \)

So \( x = \frac{1}{2}(3 + \sqrt{5}) \) as \( x \) must be greater than 1

B1 For stating or using this fact
B1 For either \( x = f(x) \) or \( x = f^{-1}(x) \)
M1 For solving the relevant quadratic equation
A1 4 For showing the given answer fully

### Question 7

**(i)** 
\[ \tan 2x = \frac{2\tan x}{1 - \tan^2 x} \]

B1 1 For correct RHS stated

**(ii)** 
\[ \frac{8t}{1-t^2} + 3x\frac{1}{t}(1+t^2) = 0 \]

B1 For \( \cot x = \frac{1}{t} \) seen
B1 For \( \sec^2 x = 1 + t^2 \) seen

Hence \( 8t^2 + 3(1-t^2)(1+t^2) = 0 \)

\( i.e. \) \( 3t^4 - 8t^2 - 3 = 0 \), as required

M1 For complete substitution in terms of \( t \)
A1 4 For showing given equation correctly

**(iii)** 
\( (3t^2 + 1)(t^2 - 3) = 0 \)

Hence \( t = \pm\sqrt{3} \)

So \( x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \)

M1 For factorising or other solution method
A1 For \( t^2 = 3 \) found correctly
A1 For any two correct angles
A1 4 For all four correct and no others
### 8

(i) \[
\frac{dy}{dx} = \frac{2 \ln x}{x}
\]

\[
\frac{d^2 y}{dx^2} = \frac{x(2/x) - 2 \ln x}{x^2} = \frac{2 - 2 \ln x}{x^2}
\]

- **M1** For relevant attempt at the chain rule
- **A1** For correct result, in any form
- **M1** For relevant attempt at quotient rule
- **A1 4** For correct simplified answer

(ii) For maximum gradient, \(2 - 2 \ln x = 0 \Rightarrow x = e\)

Hence \(P\) is \((e, 1)\)

The gradient at \(P\) is \(\frac{2}{e}\)

Tangent at \(P\) is \(y - 1 = \frac{2}{e}(x - e)\)

Hence, when \(x = 0\), \(y = -1\) as required

- **M1** For equating second derivative to zero
- **A1** For correct value \(e\)
- **A1√** For stating or using the \(y\)-coordinate
- **A1√** For stating or using the gradient at \(P\)
- **M1** For forming the equation of the tangent
- **A1 6** For correct verification of \((0, -1)\)

### 9

(i) \(a = \frac{1}{2} \pi\)

- **B1 1** For correct exact value stated

(ii) \(x = \tan(\frac{1}{2} \pi) = 1\)

- **M1** For use of \(x = \tan(\frac{1}{2} a)\)
- **A1√** 2 For correct answer, following their \(a\)

(iii)

Asymptotes are \(y = \pm 2a\)

- **B1** For \(x\)-translation of (approx) +1
- **B1** For \(y\)-stretch with (approx) factor 2
- **B1 3** For correct statement of asymptotes

\[
x = \tan^{-1} x - 2 \tan^{-1}(x - 1)
\]

(iv) \[
\begin{align*}
1.535 & \quad 0.993 & \quad 0.983 \\
1.545 & \quad 0.996 & \quad 0.998
\end{align*}
\]

Hence graphs cross between 1.535 and 1.545

- **A1 2** For correct details and explanation

(v) Relevant values of \((\tan^{-1} x)^2\) are (approximately)

\[
0, \quad 0.0600, \quad 0.2150, \quad 0.4141, \quad 0.6169
\]

\[
\frac{1}{12} \{0 + 4(0.0600 + 0.4141) + 2 \times 0.2150 + 0.6169\}
\]

Hence required approximation is 0.245

- **M1** For the relevant function values seen or implied; must be radians, not degrees
- **M1** For use of correct formula with \(h = \frac{1}{3}\)
- **A1 3** For correct (2 or 3sf) answer
OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MATHEMATICS 4724
Core Mathematics 4
Specimen Paper

Additional materials:
Answer booklet
Graph paper
List of Formulae (MF 1)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

• Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
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1 Find the quotient and remainder when \( x^4 + 1 \) is divided by \( x^2 + 1 \). [4]

2 (i) Expand \((1 - 2x)^{-2}\) in ascending powers of \(x\), up to and including the term in \(x^3\). [4]

(ii) State the set of values for which the expansion in part (i) is valid. [1]

3 Find \( \int_0^1 xe^{-2x} \, dx \), giving your answer in terms of \(e\). [5]

4

As shown in the diagram the points \(A\) and \(B\) have position vectors \(a\) and \(b\) with respect to the origin \(O\).

(i) Make a sketch of the diagram, and mark the points \(C\), \(D\) and \(E\) such that \(\overline{OC} = 2a\), \(\overline{OD} = 2a + b\) and \(\overline{OE} = \frac{1}{3} \overline{OD}\). [3]

(ii) By expressing suitable vectors in terms of \(a\) and \(b\), prove that \(E\) lies on the line joining \(A\) and \(B\). [4]

5 (i) For the curve \(2x^2 + xy + y^2 = 14\), find \(\frac{dy}{dx}\) in terms of \(x\) and \(y\). [4]

(ii) Deduce that there are two points on the curve \(2x^2 + xy + y^2 = 14\) at which the tangents are parallel to the \(x\)-axis, and find their coordinates. [4]
The diagram shows the curve with parametric equations
\[ x = a \sin \theta, \quad y = a \theta \cos \theta, \]
where \( a \) is a positive constant and \( -\pi \leq \theta \leq \pi \). The curve meets the positive \( y \)-axis at \( A \) and the positive \( x \)-axis at \( B \).

(i) Write down the value of \( \theta \) corresponding to the origin, and state the coordinates of \( A \) and \( B \). [3]

(ii) Show that \( \frac{dy}{dx} = 1 - \theta \tan \theta \), and hence find the equation of the tangent to the curve at the origin. [6]

7 The line \( L_1 \) passes through the point \((3, 6, 1)\) and is parallel to the vector \(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}\). The line \( L_2 \) passes through the point \((3, -1, 4)\) and is parallel to the vector \(\mathbf{i} - 2\mathbf{j} + \mathbf{k}\).

(i) Write down vector equations for the lines \( L_1 \) and \( L_2 \). [2]

(ii) Prove that \( L_1 \) and \( L_2 \) intersect, and find the coordinates of their point of intersection. [5]

(iii) Calculate the acute angle between the lines. [4]

8 Let \( I = \int \frac{1}{x(1+\sqrt{x})^2} \, dx \).

(i) Show that the substitution \( u = \sqrt{x} \) transforms \( I \) to \( \int \frac{2}{u(1+u)^2} \, du \). [3]

(ii) Express \( \frac{2}{u(1+u)^2} \) in the form \( \frac{A}{u} + \frac{B}{1+u} + \frac{C}{(1+u)^2} \). [5]

(ii) Hence find \( I \). [4]
A cylindrical container has a height of 200 cm. The container was initially full of a chemical but there is a leak from a hole in the base. When the leak is noticed, the container is half-full and the level of the chemical is dropping at a rate of 1 cm per minute. It is required to find for how many minutes the container has been leaking. To model the situation it is assumed that, when the depth of the chemical remaining is $x$ cm, the rate at which the level is dropping is proportional to $\sqrt{x}$.

Set up and solve an appropriate differential equation, and hence show that the container has been leaking for about 80 minutes.
1. \( \frac{x^4 + 1}{x^2 + 1} = x^2 - 1 + \frac{2}{x^2 + 1} \)

   - **B1** For correct leading term \( x^2 \) in quotient
   - **M1** For evidence of correct division process
   - **A1** For correct quotient \( x^2 - 1 \)
   - **A1** For correct remainder 2

2. (i) \((1 - 2x)^{\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-2x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(-2x)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}(-2x)^3 + \ldots \)

   \[ = 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 \]

   - **M1** For 2nd, 3rd or 4th term OK (unsimplified)
   - **A1** For \( + x \) correct
   - **A1** For \( + \frac{3}{2}x^2 \) correct
   - **A1** For \( + \frac{5}{2}x^3 \) correct

(ii) Valid for \(|x| < \frac{1}{2}\)

   - **B1** For any correct expression(s)

3. \( \int_0^1 x e^{-2x} \, dx = \left[-\frac{1}{2} x e^{-2x}\right]_0^1 - \frac{1}{2} e^{-2x} \, dx \)

   \[ = \left[-\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x}\right]_0^1 \]

   \[ = \frac{1}{4} - \frac{1}{4} e^{-2} \]

   - **M1** For attempt at ‘parts’ going the correct way
   - **A1** For correct terms \(-\frac{1}{2} x e^{-2x} - \int -\frac{1}{4} e^{-2x} \, dx \)
   - **M1** For consistent attempt at second integration
   - **M1** For correct use of limits throughout
   - **A1** For correct (exact) answer in any form

4. (i)

   - **B1** For \( C \) correctly located on sketch
   - **B1** For \( D \) correctly located on sketch
   - **B1\( \hat{\ } \)** For \( E \) correctly located wrt \( O \) and \( D \)

(ii) \( \overline{AE} = \frac{1}{2}(2a + b) - a = \frac{1}{2}(b - a) \)

   Hence \( \overline{AE} \) is parallel to \( \overline{AB} \)
   i.e. \( E \) lies on the line joining \( A \) to \( B \)

   - **M1** For relevant subtraction involving \( \overline{OE} \)
   - **A1** For correct expression for \( \pm \overline{AE} \) or \( \overline{EB} \)
   - **A1** For correct recognition of parallel property
   - **A1** For complete proof of required result

5. (i) \( 4x + x \frac{dy}{dx} + y + 2y \frac{dy}{dx} = 0 \)

   Hence \( \frac{dy}{dx} = -\frac{4x + y}{x + 2y} \)

   - **B1** For correct terms \( \frac{dy}{dx} + y \)
   - **B1** For correct term \( 2y \frac{dy}{dx} \)
   - **M1** For solving for \( \frac{dy}{dx} \)
   - **A1** For any correct form of expression

(ii) \( \frac{dy}{dx} = 0 \Rightarrow y = -4x \)

   Hence \( 2x^2 + (-4x^2) + (-4x)^2 = 14 \)
   i.e. \( x^2 = 1 \)

   So the two points are \((1, -4)\) and \((-1, 4)\)

   - **M1** For stating or using their \( \frac{dy}{dx} = 0 \)
   - **M1** For solving simultaneously with curve equn
   - **A1** For correct value of \( x^2 \) (or \( y^2 \))
   - **A1** For both correct points identified
### Question 6

**Part (i)**

\[ \theta = 0 \text{ at the origin} \]

A is \((0, a\pi)\)

B is \((a, 0)\)

**Marks:** B1

- For the correct value
- For the correct y-coordinate at A
- For the correct x-coordinate at B

**Part (ii)**

\[ \frac{dx}{d\theta} = a\cos\theta \]

\[ \frac{dy}{d\theta} = a(\cos\theta - \theta\sin\theta) \]

Hence

\[ \frac{dy}{dx} = \frac{\cos\theta - \theta\sin\theta}{\cos\theta} = 1 - \theta\tan\theta \]

Gradient of tangent at the origin is 1

Hence equation is \(y = x\)

**Marks:** B1

- For correct differentiation of \(x\)
- For differentiating \(y\) using product rule
- For use of \(\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{d\theta}{dx}\)
- For given result correctly obtained
- For using \(\theta = 0\)
- For correct equation

### Question 8

**Part (i)**

\[ I = \int \frac{1}{u^2(1+u)^2} \times 2u \, du = \int \frac{2}{u(1+u)^2} \, du \]

**Marks:** M1

- For any attempt to find \(\frac{dx}{du}\) or \(\frac{du}{dx}\)
- For showing the given result correctly

**Part (ii)**

\[ 2 = A(1+u)^2 + Bu(1+u) + Cu \]

- \(A = 2\)
- \(C = -2\)
- \(0 = A + B\) (e.g.)
- \(B = -2\)

**Marks:** M1

- For correct identity stated
- For correct value stated
- For correct value stated
- For any correct equation involving \(B\)
- For correct value

**Part (iii)**

\[ 2\ln u - 2\ln(1+u) + \frac{2}{1+u} \]

Hence

\[ I = \ln x - 2\ln(1+\sqrt{x}) + \frac{2}{1+\sqrt{x}} + c \]

**Marks:** B1

- For \(A\ln u + B\ln(1+u)\) with their values
- For \(-C(1+u)^{-1}\) with their value
- For substituting back
- For completely correct answer (excluding \(c\)}
\[
\frac{dx}{dt} = -k\sqrt{x}
\]

\(x = 100\) and \(\frac{dx}{dt} = -1 \Rightarrow k = 0.1\)

Hence equation is \(\frac{dx}{dt} = -0.1\sqrt{x}\)

\[\int x^{-\frac{1}{2}} \, dx = -0.1 \int dt \Rightarrow 2x^{\frac{1}{2}} = -0.1t + c\]

\[x = 200, t = 0 \Rightarrow c = 2\sqrt{200}\]

So when \(x = 100, \quad 2\sqrt{100} = -0.1t + 2\sqrt{200}\)

i.e. \(t = 82.8\)
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- You are reminded of the need for clear presentation in your answers.
1 Use formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that
\[
\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2).
\] [5]

2 The cubic equation $x^3 - 6x^2 + kx + 10 = 0$ has roots $p - q$, $p$ and $p + q$, where $q$ is positive.

(i) By considering the sum of the roots, find $p$. [2]

(ii) Hence, by considering the product of the roots, find $q$. [3]

(iii) Find the value of $k$. [3]

3 The complex number $2 + i$ is denoted by $z$, and the complex conjugate of $z$ is denoted by $z^\ast$.

(i) Express $z^2$ in the form $x + iy$, where $x$ and $y$ are real, showing clearly how you obtain your answer. [2]

(ii) Show that $4z - z^2$ simplifies to a real number, and verify that this real number is equal to $zz^\ast$. [3]

(iii) Express $\frac{z + 1}{z - 1}$ in the form $x + iy$, where $x$ and $y$ are real, showing clearly how you obtain your answer. [3]

4 A sequence $u_1, u_2, u_3, \ldots$ is defined by
\[
u_n = 3^{2n} - 1.
\]

(i) Write down the value of $u_1$. [1]

(ii) Show that $u_{n+1} - u_n = 8 \times 3^{2n}$. [3]

(iii) Hence prove by induction that each term of the sequence is a multiple of 8. [4]
5  (i) Show that  
\[
\frac{1}{2r-1} - \frac{1}{2r+1} = \frac{2}{4r^2-1}.
\]

(ii) Hence find an expression in terms of \( n \) for  
\[
\frac{2}{3} + \frac{2}{15} + \frac{2}{35} + \ldots + \frac{2}{4n^2-1}.
\]

(iii) State the value of  
(a) \[
\sum_{r=1}^{\infty} \frac{2}{4r^2-1},
\]

(b) \[
\sum_{r=n+1}^{\infty} \frac{2}{4r^2-1}.
\]

6  In an Argand diagram, the variable point \( P \) represents the complex number \( z = x + iy \), and the fixed point \( A \) represents \( a = 4 - 3i \).

(i) Sketch an Argand diagram showing the position of \( A \), and find \( |a| \) and \( \arg a \).

(ii) Given that \( |z - a| = |a| \), sketch the locus of \( P \) on your Argand diagram.

(iii) Hence write down the non-zero value of \( z \) corresponding to a point on the locus for which

(a) the real part of \( z \) is zero,

(b) \( \arg z = \arg a \).

7  The matrix \( A \) is given by  
\[
A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.
\]

(i) Draw a diagram showing the unit square and its image under the transformation represented by \( A \).

(ii) The value of \( \det A \) is 5. Show clearly how this value relates to your diagram in part (i).

A represents a sequence of two elementary geometrical transformations, one of which is a rotation \( R \).

(iii) Determine the angle of \( R \), and describe the other transformation fully.

(iv) State the matrix that represents \( R \), giving the elements in an exact form.
The matrix $M$ is given by $M = \begin{pmatrix} a & 2 & -1 \\ 2 & 3 & -1 \\ 2 & -1 & 1 \end{pmatrix}$, where $a$ is a constant.

(i) Show that the determinant of $M$ is $2a$. \[2\]

(ii) Given that $a \neq 0$, find the inverse matrix $M^{-1}$. \[4\]

(iii) Hence or otherwise solve the simultaneous equations

\begin{align*}
    x + 2y - z &= 1, \\
    2x + 3y - z &= 2, \\
    2x - y + z &= 0.
\end{align*}

\[3\]

(iv) Find the value of $k$ for which the simultaneous equations

\begin{align*}
    2y - z &= k, \\
    2x + 3y - z &= 2, \\
    2x - y + z &= 0,
\end{align*}

have solutions. \[3\]

(v) Do the equations in part (iv), with the value of $k$ found, have a solution for which $x = z$? Justify your answer. \[2\]
1 \[ \sum_{r=1}^{n} r(r+1) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r = \frac{1}{6} n(n+1)(2n+1) + \frac{1}{2} n(n+1) \]
\[ = \frac{1}{3} n(n+1)(n+2) \]

M1 For considering the two separate sums
A1 For either correct sum formula stated
A1 For completely correct expression
M1 For factorising attempt
A1 5 For showing given answer correctly

2 (i) \[(p-q) + p + (p+q) = 6 \Rightarrow p = 2\]
M1 For use of \(\Sigma x = -b/a\)
A1 2 For correct answer

(ii) \[2(2-q)(2+q) = -10\]
Hence \[4 - q^2 = -5 \Rightarrow q = 3\]
B1  For use of \(\alpha\beta\gamma = -d/a\)
M1 For expanding and solving for \(q^2\)
A1 3 For correct answer

(iii) EITHER: Roots are \(-1, 2, 5\)
\[-1 \times 2 \times 2 \times 5 = -k\]
i.e. \(k = 3\)
B1  For stating or using three numerical roots
M1 For use of \(\Sigma \alpha\beta\gamma = c/a\)
A1  For correct answer from their roots

OR: Roots are \(-1, 2, 5\)
Equation is \((x + 1)(x - 2)(x - 5) = 0\)
Hence \(k = 3\)
B1  For stating or using three numerical roots
M1 For stating and expanding factorised form
A1 3 For correct answer from their roots

3 (i) \[z^2 = (2+i)^2 = 4+4i + i^2 = 3+4i\]
M1 For showing 3-term or 4-term expansion
A1 2 For correct answer

(ii) \[4z - z^2 = 8+4i - 3 - 4i = 5\]
\[zz^* = (2+i)(2-i) = 5\]
B1 For correct value 5
B1 For stating or using \(z^* = 2 - i\)
B1 3 For correct verification of given result

(iii) \[\frac{z+1}{z-1} = 3 + i \quad \frac{3+i}{1+i} = \frac{4-2i}{2} = 2-i\]
B1 For correct initial form \(\frac{3+i}{1+i}\)
M1 For multiplying top and bottom by \(1-i\)
A1 3 For correct answer \(2-i\)

4 (i) \[u_t = 8\]
B1 1 For correct value stated

(ii) \[3^{2(n+1)} - 1 - (3^{2n} - 1) = 9 \times 3^{2n} - 3^{2n} = 8 \times 3^{2n}\]
B1 For stating or using \(u_{n+1} = 3^{2(n+1)} - 1\)
M1 For relevant manipulation of indices in \(u_{n+1}\)
A1 3 For showing given answer correctly

(iii) \(u_t\) is divisible by 8, from (i)
Suppose \(u_k\) is divisible by 8, i.e. \(u_k = 8a\)
Then \(u_{k+1} = u_k + 8 \times 3^{2k} = 8(a + 3^{2k}) = 8b\)
i.e. \(u_{k+1}\) is also divisible by 8, and result follows by the induction principle
B1 For explicit check for \(u_t\)
M1 For induction hypothesis \(u_k\) is mult. of 8
M1 For obtaining and simplifying expr. for \(u_{k+1}\)
A1 4 For correct conclusion, stated and justified
5 
(i) \[ \text{LHS} = \frac{2r+1-(2r-1)}{(2r-1)(2r+1)} = \frac{2}{4r^2-1} = \text{RHS} \] 
M1 For correct process for adding fractions 
A1 2 For showing given result correctly 

(ii) Sum is \[ \left( \frac{1}{1} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \cdots + \left( \frac{1}{2n-1} - \frac{1}{2n+1} \right) \] 
M1 For expressing terms as differences using (i) 
A1 For at least first two and last terms correct 
M1 For cancelling pairs of terms 
A1 4 For any correct form 

(iii) (a) Sum to infinity is \( 1 \) B1 For correct value; follow their (ii) if convergent 

(b) Required sum is \( \frac{1}{2n+1} \) B1 For correct difference of their (iii)(a) and (ii) 

6 
(i) (See diagram in part (ii) below) \[ |a| = \sqrt{(3^2 + 4^2)} = 5 \] \[ \arg a = -\tan^{-1} \left( \frac{4}{3} \right) = -0.644 \] B1 For point A correctly located 
B1 For correct value for the modulus 
M1 For any correct relevant trig statement 
A1 4 For correct answer (radians or degrees) 

(ii) [Diagram] B1 For any indication that locus is a circle 
B1 For any indication that the centre is at A 
B1 3 For a completely correct diagram 

(iii) (a) \( z = -6i \) B1 1 For correct answer 

(b) \( z = 8 - 6i \) M1 For identification of end of diameter thru A 
A1 2 For correct answer 

7 
(i) \[ \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -2 & -1 \\ 0 & 2 & 1 & 3 \end{pmatrix} \] 
M1 For at least one correct image 
A1 For all vertices correct 
A1 3 For correct diagram 

(ii) The area scale-factor is 5 
The transformed square has side of length \( \sqrt{5} \) 
So its area is 5 times that of the unit square B1 For identifying det as area scale factor 
M1 For calculation method relating to large sq. 
A1 3 For a complete explanation 

(iii) Angle is \( \tan^{-1}(2) = 63.4^\circ \) B1 For \( \tan^{-1}(2) \), or equivalent 
Enlargement with scale factor \( \sqrt{5} \) B1 For stating ‘enlargement’ 
B1 3 For correct (exact) scale factor 

(iv) \[ \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \] 
M1 For correct \( \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \) pattern 
A1 2 For correct matrix in exact form 

8 
9 
10 
11
(i) \[ \text{det } \mathbf{M} = a(3-1) - 2(2-(2)) - 1(-2 - 6) \]
\[ = 2a \]
For correct expansion process

\[ a = -\frac{M}{M_1} \]
For showing given answer correctly

(ii) \[ \mathbf{M}^{-1} = \frac{1}{2a} \begin{pmatrix} 2 & -1 & 1 \\ -4 & a+2 & a-2 \\ 8 & a+4 & 3a-4 \end{pmatrix} \]
For correct process for adjoint entries

\[ a = \frac{A_1}{2} \]
For showing given answer correctly

\[ \frac{1}{2a} \begin{pmatrix} 2 & -1 & 1 \\ -4 & a+2 & a-2 \\ 8 & a+4 & 3a-4 \end{pmatrix} \]
For at least 4 correct entries in adjoint

\[ A_1 \]
For dividing by the determinant

\[ \frac{1}{2a} \begin{pmatrix} 2 & -1 & 1 \\ -4 & a+2 & a-2 \\ 8 & a+4 & 3a-4 \end{pmatrix} \]
For completely correct inverse

(iii) \[ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{M}^{-1} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} , \text{ with } a = 1 \]
For correct statement involving inverse

\[ x = 0, y = 1, z = 1 \]
For carrying out the correct multiplication

A1 3 For all three correct values

(iv) Eliminating \( x \) gives \( 4y - 2z = 2 \)
For eliminating \( x \) from 2nd and 3rd eqns

So for consistency with 1st eqn, \( k = 1 \)
For comparing two \( y-z \) equations

A1 3 For correct value for \( k \)

(v) Solving \( x + 3y = 2, 3x - y = 0 \) gives \( x = \frac{1}{2}, y = \frac{3}{2} \)
For using \( x = z \) to solve a pair of eqns

These values check in \( 2y - x = 1 \), so soln exists
For a completely correct demonstration

A1 2 For a completely correct demonstration
Oxford Cambridge and RSA Examinations

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

Mathematics

Further Pure Mathematics 2

Specimen Paper

Additional materials:
- Answer booklet
- Graph paper
- List of Formulae (MF 1)

Time 1 hour 30 minutes

Instructions to Candidates
- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer all the questions.
- Give non-exact numerical answers correct to 3 significant figures, unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphic calculator in this paper.

Information for Candidates
- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.
- You are reminded of the need for clear presentation in your answers.
1 (i) Starting from the definition of \( \cosh x \) in terms of \( e^x \), show that \( \cosh 2x = 2\cosh^2 x - 1 \). [2]

(ii) Given that \( \cosh 2x = k \), where \( k > 1 \), express each of \( \cosh x \) and \( \sinh x \) in terms of \( k \). [4]

2

![Graph of the equation \( y = \frac{2x^2 + 3x + 3}{x + 1} \).]

The diagram shows the graph of

\[ y = \frac{2x^2 + 3x + 3}{x + 1}. \]

(i) Find the equations of the asymptotes of the curve. [3]

(ii) Prove that the values of \( y \) between which there are no points on the curve are \(-5\) and \(3\). [4]

3 (i) Find the first three terms of the Maclaurin series for \( \ln(2 + x) \). [4]

(ii) Write down the first three terms of the series for \( \ln(2 - x) \), and hence show that, if \( x \) is small, then

\[
\ln\left(\frac{2 + x}{2 - x}\right) = x. \] [3]
4 The equation of a curve, in polar coordinates, is
\[ r = 2 \cos 2\theta \quad (-\pi < \theta \leq \pi). \]

(i) Find the values of \( \theta \) which give the directions of the tangents at the pole. [3]

One loop of the curve is shown in the diagram.

(ii) Find the exact value of the area of the region enclosed by the loop. [5]

5

The diagram shows the curve \( y = \frac{1}{x+1} \) together with four rectangles of unit width.

(i) Explain how the diagram shows that
\[ \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} < \int_{0}^{4} \frac{1}{x+1} \, dx. \] [2]

The curve \( y = \frac{1}{x+2} \) passes through the top left-hand corner of each of the four rectangles shown.

(ii) By considering the rectangles in relation to this curve, write down a second inequality involving \( \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \) and a definite integral. [2]

(iii) By considering a suitable range of integration and corresponding rectangles, show that
\[ \ln(500.5) < \sum_{r=2}^{1000} \frac{1}{r} < \ln(1000). \] [4]
6  (i) Given that \( I_n = \int_0^1 x^n \sqrt{(1 - x)} \, dx \), prove that, for \( n \geq 1 \),
\[
(2n + 3)I_n = 2nI_{n-1}.
\] [6]

(ii) Hence find the exact value of \( I_2 \). [4]

7  The curve with equation
\[
y = \frac{x}{\cosh x}
\]
has one stationary point for \( x > 0 \).

(i) Show that the \( x \)-coordinate of this stationary point satisfies the equation \( x \tanh x - 1 = 0 \). [2]

The positive root of the equation \( x \tanh x - 1 = 0 \) is denoted by \( \alpha \).

(ii) Draw a sketch showing (for positive values of \( x \)) the graph of \( y = \tanh x \) and its asymptote, and the
graph of \( y = \frac{1}{x} \). Explain how you can deduce from your sketch that \( \alpha > 1 \). [3]

(iii) Use the Newton-Raphson method, taking first approximation \( x_1 = 1 \), to find further approximations
\( x_2 \) and \( x_3 \) for \( \alpha \). [5]

(iv) By considering the approximate errors in \( x_1 \) and \( x_2 \), estimate the error in \( x_3 \). [3]

8  (i) Use the substitution \( t = \tan \frac{1}{2}x \) to show that
\[
\int_0^{\frac{1}{2}\pi} \sqrt{\frac{1 - \cos x}{1 + \sin x}} \, dx = 2 \sqrt{2} \int_0^1 \frac{t}{(1 + t)(1 + t^2)} \, dt.
\] [4]

(ii) Express \( \frac{t}{(1 + t)(1 + t^2)} \) in partial fractions. [5]

(iii) Hence find \( \int_0^{\frac{1}{2}\pi} \sqrt{\frac{1 - \cos x}{1 + \sin x}} \, dx \), expressing your answer in an exact form. [4]
1 (i) \[ \text{RHS} = 2\left(\frac{1}{2}(e^x + e^{-x})\right)^2 - 1 = \frac{1}{2}(e^{2x} + e^{-2x}) = \text{LHS} \]

\[ M1 \quad \text{For correct squaring of } (e^x + e^{-x}) \]

\[ A1 \quad \text{For completely correct proof} \]

(ii) \[ 2\cosh^2 x - 1 = k \Rightarrow \cosh x = \sqrt{\frac{1}{2}(1 + k)} \quad M1 \quad \text{For use of (i) and solving for } \cosh x \]

\[ 2\sinh^2 x + 1 = k \Rightarrow \sinh x = \pm \sqrt{\frac{1}{2}(k - 1)} \quad A1 \quad \text{For correct positive square root only} \]

\[ A1 \quad \text{For use of } \cosh^2 x - \sinh^2 x = 1, \text{ or equivalent} \]

\[ A1 \quad \text{For both correct square roots} \]

2 (i) \( x = -1 \) is an asymptote

\[ y = 2x + 1 + \frac{2}{x+1} \]

\[ \text{Hence } y = 2x + 1 \text{ is an asymptote} \]

\[ B1 \quad \text{For correct equation of vertical asymptote} \]

\[ M1 \quad \text{For algebraic division, or equivalent} \]

\[ A1 \quad \text{For correct equation of oblique asymptote} \]

(ii) **EITHER:** Quadratic \( 2x^2 + (3 - y)x + (3 - y) = 0 \)

\[ \text{has no real roots if } (3 - y)^2 < 8(3 - y) \]

\[ \text{Hence } (3 - y)(-5 - y) < 0 \]

\[ \text{So required values are 3 and -5} \]

\[ M1 \quad \text{For using discriminant of relevant quadratic} \]

\[ A1 \quad \text{For correct inequality or equation in } y \]

\[ M1 \quad \text{For factorising, or equivalent} \]

\[ A1 \quad \text{For given answer correctly shown} \]

**OR:**

\[ \frac{dy}{dx} = 2 - \frac{2}{(x+1)^2} = 0 \]

\[ \text{Hence } (x+1)^2 = 1 \]

\[ \text{So } x = -2 \text{ and } 0 \Rightarrow y = -5 \text{ and 3} \]

\[ M1 \quad \text{For differentiating and equating to zero} \]

\[ A1 \quad \text{For correct simplified quadratic in } x \]

\[ M1 \quad \text{For solving for } x \text{ and substituting to find } y \]

\[ A1 \quad \text{For given answer correctly shown} \]

3 (i) **EITHER:** If \( f(x) = \ln(x+2) \), then \( f'(x) = \frac{1}{2+x} \)

and \( f''(x) = -\frac{1}{(2+x)^2} \)

\[ f(0) = \ln 2, f'(0) = \frac{1}{2}, \quad f''(0) = -\frac{1}{4} \]

\[ \text{Hence } \ln(x+2) = \ln 2 + \frac{1}{2}x - \frac{1}{8}x^2 + ... \]

\[ M1 \quad \text{For at least one differentiation attempt} \]

\[ A1 \quad \text{For correct first and second derivatives} \]

\[ A1 \quad \text{For all three evaluations correct} \]

\[ A1 \quad \text{For three correct terms} \]

**OR:**

\[ \ln(2+x) = \ln[2(1+\frac{1}{2}x)] \]

\[ = \ln 2 + \ln(1+\frac{1}{2}x) \]

\[ = \ln 2 + \frac{1}{2}x - \frac{1}{8}x^2 + ... \]

\[ A1 \quad \text{For factorising in this way} \]

\[ A1 \quad \text{For using relevant log law correctly} \]

\[ M1 \quad \text{For use of standard series expansion} \]

\[ A1 \quad \text{For three correct terms} \]

(ii) \( \ln(2-x) = \ln 2 - \frac{1}{2}x - \frac{1}{8}x^2 \)

\[ \ln\left(\frac{2+x}{2-x}\right) = (\ln 2 + \frac{1}{2}x - \frac{1}{8}x^2) - (\ln 2 - \frac{1}{2}x - \frac{1}{8}x^2) \]

\[ = x, \text{ as required} \]

\[ B1 \quad \text{For replacing } x \text{ by } -x \]

\[ M1 \quad \text{For subtracting the two series} \]

\[ A1 \quad \text{For showing given answer correctly} \]
### Question 4

(i) \( r = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow \theta = \pm \frac{1}{2} \pi, \pm \frac{3}{2} \pi \)

M1 For equating \( r \) to zero and solving for \( \theta \)
A1 For any two correct values
A1 3 For all four correct values and no others

(ii) Area is \( \frac{1}{4} \int_{-\pi}^{\pi} 4 \cos^2 2\theta \, d\theta \)

M1 For us of correct formula \( \frac{1}{2} \int r^2 \, d\theta \)
B1 For correct limits from (i)
A1 For using double-angle formula
A1 5 For correct (exact) answer

### Question 5

(i) LHS is the total area of the four rectangles

RHS is the corresponding area under the curve, which is clearly greater

B1 For identifying rectangle areas (not heights)
B1 2 For correct explanation

(ii) \( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{3} > \int_{0}^{4} \frac{1}{x + 2} \, dx \)

M1 For attempt at relevant new inequality
A1 2 For correct statement

(iii) Sum is the area of 999 rectangles

Bounds are \( \int_{0}^{999} \frac{1}{x + 2} \, dx \) and \( \int_{0}^{999} \frac{1}{x + 1} \, dx \)

So lower bound is \( \ln(x + 2)_{0}^{999} = \ln(500.5) \)

and upper bound is \( \ln(x + 1)_{0}^{999} = \ln(1000) \)

M1 For considering the sum as an area again
M1 For stating either integral as a bound
A1 For showing the given value correctly
A1 4 Ditto

### Question 6

(i) \( I_n = \left[ -\frac{2}{3} x^n (1 - x)^2 \right]_0^1 + \frac{2}{3} n \int_{0}^{1} x^{n-1} (1 - x)^2 \, dx \)

M1 For using integration by parts
A1 For correct first stage result
M1 For use of limits in integrated term
A1 For splitting the remaining integral up
A1 For correct relation between \( I_n \) and \( I_{n-1} \)
A1 6 For showing given answer correctly

Hence \( 2n + 3)I_n = 2nI_{n-1} \), as required

(ii) \( I_2 = \frac{3}{4} I_1 = \frac{3}{4} \times \frac{2}{3} I_0 \)

M1 For two uses of the recurrence relation
A1 For correct expression in terms of \( I_0 \)

Hence \( I_2 = \frac{3}{4} \left[ -\frac{2}{3} (1 - x)^2 \right]_0^1 = \frac{16}{105} \)

M1 For evaluation of \( I_0 \)
A1 4 For correct answer
### Q7

#### (i)
\[ \frac{dy}{dx} = \frac{\cosh x - x \sinh x}{\cosh^2 x} \]
Max occurs when \( \cosh x = \sinh x \), i.e. \( \tanh x = 1 \)
- **M1** For differentiating and equating to zero
- **A1** For showing given result correctly

#### (ii)
\[ y = \frac{x - \tanh x}{\tanh x + x \sech^2 x} \]
- **B1** For correct sketch of \( y = \tanh x \)
- **B1** For identification of asymptote \( y = 1 \)
- **B1** For correct explanation of \( \alpha > 1 \) based on intersection \( (1, 1) \) of \( y = 1/x \) with \( y = 1 \)

#### (iii)
\[ x_{n+1} = x_n - \frac{x_n \tanh x_n - 1}{\tanh x_n + x_n \sech^2 x_n} \]
- **M1** For correct Newton-Raphson structure
- **A1** For all details in \( x - \frac{f(x)}{f'(x)} \) correct
- **A1** For using Newton-Raphson at least once
- **A1** For \( x_2 \) correct to at least 3sf
- **A1** For \( x_3 \) correct to at least 4sf

#### (iv)
\[ e_1 = 0.2, \ e_2 = -0.002 \]
\[ \frac{e_1}{e_2} = \frac{e_2}{e_1} \Rightarrow e_3 = -2 \times 10^{-7} \]
- **B1\(^\checkmark\)** For both magnitudes correct
- **M1** For use of quadratic convergence property
- **A1** For answer of correct magnitude

### Q8

#### (i)
\[ \frac{dt}{dx} = \frac{1}{2}(1 + t^2) \]
\[ \int_0^{\pi/2} \frac{1 - \cos x}{\sqrt{1 + \sin x}} \, dx = \int_0^1 \frac{1 - \frac{1 - t^2}{1 + t^2}}{1 + \frac{2t}{1 + t^2}} \, \frac{2}{1 + t^2} \, dt \]
\[ = \int_0^1 \frac{2t^2}{(1 + t^2)^2} \, dt = 2\sqrt{2} \int_0^1 \frac{t}{(1 + t)(1 + t^2)} \, dt \]
- **B1** For this relation, stated or used
- **M1** For complete substitution for \( x \) in integrand
- **B1** For justification of limits 0 and 1 for \( t \)
- **A1** For correct simplification to given answer

#### (ii)
\[ t = \frac{A(1 + t^2)}{1 + t} + Ct \]
\[ \text{Hence } t = A(1 + t^2) + (Bt + C)(1 + t) \]
\[ \text{From which } A = -\frac{1}{2}, \ B = \frac{1}{2}, \ C = \frac{1}{2} \]
- **B1** For statement of correct form of pfs
- **M1** For any use of the identity involving \( B \) or \( C \)
- **B1** For correct value of \( A \)
- **A1** For correct value of \( B \)
- **A1** For correct value of \( C \)

#### (iii)
\[ \text{Int is } 2\sqrt{2} \left[ -\frac{1}{2} \ln(1 + t) + \frac{1}{4} \ln(1 + t^2) + \frac{1}{2} \tan^{-1} t \right]_0^1 \]
\[ = \frac{1}{4}(\pi - 2\ln 2) \sqrt{2} \]
- **B1\(^\checkmark\)** For both logarithm terms correct
- **B1\(^\checkmark\)** For the inverse tan term correct
- **M1** For use of appropriate limits
- **A1** For correct (exact) answer in any form
INSTRUCTIONS TO CANDIDATES

• Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
• Answer all the questions.
• Give non-exact numerical answers correct to 3 significant figures, unless a different degree of accuracy is specified in the question or is clearly appropriate.
• You are permitted to use a graphic calculator in this paper.

INFORMATION FOR CANDIDATES

• The number of marks is given in brackets [ ] at the end of each question or part question.
• The total number of marks for this paper is 72.
• Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.
• You are reminded of the need for clear presentation in your answers.
1. Find the general solution of the differential equation
\[ \frac{dy}{dx} - \frac{y}{x} = x, \]
giving y in terms of x in your answer. [5]

2. The set \( S = \{ a, b, c, d \} \) under the binary operation \( \ast \) forms a group \( G \) of order 4 with the following operation table.

<table>
<thead>
<tr>
<th>( \ast )</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>d</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>a</td>
</tr>
<tr>
<td>d</td>
<td>c</td>
<td>d</td>
<td>a</td>
<td>b</td>
</tr>
</tbody>
</table>

(i) Find the order of each element of \( G \). [3]

(ii) Write down a proper subgroup of \( G \). [1]

(iii) Is the group \( G \) cyclic? Give a reason for your answer. [1]

(iv) State suitable values for each of \( a, b, c \) and \( d \) in the case where the operation \( \ast \) is multiplication of complex numbers. [1]

3. The planes \( \Pi_1 \) and \( \Pi_2 \) have equations \( r.(i-2j+2k) = 1 \) and \( r.(2i+2j-k) = 3 \) respectively. Find

(i) the acute angle between \( \Pi_1 \) and \( \Pi_2 \), correct to the nearest degree, [4]

(ii) the equation of the line of intersection of \( \Pi_1 \) and \( \Pi_2 \), in the form \( r = a + tb \). [4]

4. In this question, give your answers exactly in polar form \( re^{i\theta} \), where \( r > 0 \) and \(-\pi < \theta \leq \pi\).

(i) Express \( 4((\sqrt{3})-i) \) in polar form. [2]

(ii) Find the cube roots of \( 4((\sqrt{3})-i) \) in polar form. [4]

(iii) Sketch an Argand diagram showing the positions of the cube roots found in part (ii). Hence, or otherwise, prove that the sum of these cube roots is zero. [3]

5. The lines \( l_1 \) and \( l_2 \) have equations

\[ \frac{x-5}{1} = \frac{y-1}{-1} = \frac{z-5}{-2} \quad \text{and} \quad \frac{x-1}{-4} = \frac{y-11}{-14} = \frac{z-2}{2}. \]

(i) Find the exact value of the shortest distance between \( l_1 \) and \( l_2 \). [5]

(ii) Find an equation for the plane containing \( l_1 \) and parallel to \( l_2 \) in the form \( ax + by + cz = d \). [4]
6 The set $S$ consists of all non-singular $2 \times 2$ real matrices $A$ such that $AQ = QA$, where

$$Q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$ 

(i) Prove that each matrix $A$ must be of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$. [4]

(ii) State clearly the restriction on the value of $a$ such that $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ is in $S$. [1]

(iii) Prove that $S$ is a group under the operation of matrix multiplication. (You may assume that matrix multiplication is associative.) [5]

7 (i) Prove that if $z = e^{i\theta}$, then $z^n + \frac{1}{z^n} = 2\cos n\theta$. [2]

(ii) Express $\cos \theta$ in terms of cosines of multiples of $\theta$, and hence find the exact value of

$$\int_0^{\pi/2} \cos^6 \theta \, d\theta.$$ [8]

8 (i) Find the value of the constant $k$ such that $y = kx^2e^{-2x}$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 2e^{-2x}. \quad [4]$$

(ii) Find the solution of this differential equation for which $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$. [7]

(iii) Use the differential equation to determine the value of $\frac{d^2y}{dx^2}$ when $x = 0$. Hence prove that $0 < y \leq 1$ for $x \geq 0$. [4]
OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MATHEMATICS

Further Pure Mathematics 3

MARK SCHEME

Specimen Paper

MAXIMUM MARK | 72
1. Integrating factor is \( e^{\int -x^{-1} \, dx} = e^{-\ln x} = \frac{1}{x} \)

\[
\frac{d}{dx} \left( \frac{y}{x} \right) = 1 \Rightarrow \int 1 \, dx = y = x^2 + cx
\]

<table>
<thead>
<tr>
<th>Question</th>
<th>Marks</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>M1</td>
<td>For finding integrating factor</td>
</tr>
<tr>
<td>1.2</td>
<td>A1</td>
<td>For correcting simplified form</td>
</tr>
<tr>
<td>1.3</td>
<td>M1</td>
<td>For using integrating factor correctly</td>
</tr>
<tr>
<td>1.4</td>
<td>B1</td>
<td>For arbitrary constant introduced correctly</td>
</tr>
<tr>
<td>1.5</td>
<td>A1</td>
<td>For correct answer in required form</td>
</tr>
</tbody>
</table>

2. (i) \( b \) is the identity and so has order 1
\[ d \ast d = b, \text{ so } d \text{ has order } 2 \]
\[ a \ast a = c \ast c = d, \text{ so } a \text{ and } c \text{ each have order } 4 \]

(ii) \( \{ b, d \} \)

(iii) \( G \) is cyclic because it has an element of order 4

(iv) \( b = 1, d = -1, a = i, c = -i \) (or vice versa for \( a, c \))

<table>
<thead>
<tr>
<th>Question</th>
<th>Marks</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>B1</td>
<td>For identifying ( b ) as the identity element</td>
</tr>
<tr>
<td>2.2</td>
<td>B1</td>
<td>For stating the order of ( d ) is 2</td>
</tr>
<tr>
<td>2.3</td>
<td>B1</td>
<td>For both orders stated</td>
</tr>
<tr>
<td>2.4</td>
<td>B1</td>
<td>For stating this subgroup</td>
</tr>
<tr>
<td>2.5</td>
<td>B1</td>
<td>For correct answer with justification</td>
</tr>
<tr>
<td>2.6</td>
<td>B1</td>
<td>For all four correct values</td>
</tr>
</tbody>
</table>

3. (i) Normals are \( i - 2j + 2k \) and \( 2i + 2j - k \)

Acute angle is \( \cos^{-1} \left( \frac{[2 - 4 - 2]}{3 \times 3} \right) = 64^\circ \)

(ii) Direction of line is \( (i - 2j + 2k) \times (2i + 2j - k) \),
\[ i.e. \, -2i + 5j + 6k \]
\[ x - 2y + 2z = 1, 2x + 2y - z = 3 \Rightarrow 3x + z = 4 \]

Hence line is \( r = i + j + k + t(-2i + 5j + 6k) \)

<table>
<thead>
<tr>
<th>Question</th>
<th>Marks</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>B1</td>
<td>For identifying both normal vectors</td>
</tr>
<tr>
<td>3.2</td>
<td>M1</td>
<td>For using the scalar product of the normals</td>
</tr>
<tr>
<td>3.3</td>
<td>M1</td>
<td>For completely correct process for the angle</td>
</tr>
<tr>
<td>3.4</td>
<td>A1</td>
<td>For correct answer</td>
</tr>
</tbody>
</table>

4. (i) \( 4(\sqrt{3} - i) = 8e^{-\frac{i}{6}\pi} \)

(ii) One cube root is \( 2e^{-\frac{i}{3}\pi} \)

Others are found by multiplying by \( e^{\pm \frac{i}{3}\pi} \)

Giving \( 2e^{\frac{i}{3}\pi} \) and \( 2e^{-\frac{i}{3}\pi} \)

(iii)

<table>
<thead>
<tr>
<th>Question</th>
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<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>B1</td>
<td>For ( r = 8 )</td>
</tr>
<tr>
<td>4.2</td>
<td>B1</td>
<td>For ( \theta = -\frac{1}{6}\pi )</td>
</tr>
<tr>
<td>4.3</td>
<td>B1^</td>
<td>For modulus and argument both correct</td>
</tr>
<tr>
<td>4.4</td>
<td>M1</td>
<td>For multiplication by either cube root of 1 (or equivalent use of symmetry)</td>
</tr>
<tr>
<td>4.5</td>
<td>A1</td>
<td>For either one of these roots</td>
</tr>
<tr>
<td>4.6</td>
<td>A1</td>
<td>For both correct</td>
</tr>
</tbody>
</table>

5. The roots have equal modulus and args differing by \( \frac{2}{3}\pi \), so adding them geometrically makes a closed equilateral triangle; i.e. sum is zero

<table>
<thead>
<tr>
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<th>Marks</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1</td>
<td>B1^</td>
<td>For correct diagram from their (ii)</td>
</tr>
<tr>
<td>5.2</td>
<td>M1</td>
<td>For geometrical interpretation of addition</td>
</tr>
<tr>
<td>5.3</td>
<td>A1</td>
<td>For a correct proof (or via components, etc)</td>
</tr>
</tbody>
</table>
### 5 (i) 
\((i-j-2k)\times(-4i-14j+2k) = -30i+6j-18k\)  
So common perp is parallel to \(5i-j+3k\)  
\((5i+j+5k) - (i+11j+2k) = 4i-10j+3k\)  
\(d = \frac{(4i-10j+3k) \times (5i-j+3k)}{|5i-j+3k|} = \frac{39}{\sqrt{35}}\)

For vector product of direction vectors  
A1 For correct vector for common perp  
B1 For calculating the difference of positions  
M1 For calculation of the projection  
A1 For correct exact answer

(ii) Normal vector for plane is \(5i-j+3k\)  
Point on plane is \(5i+j+5k\)  
Equation is \(5x-y+3z = 25-1+15\)  
i.e. \(5x-y+3z = 39\)

For correct exact answer

### 6 (i) \(AQ = QA\)  
\[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\)

i.e. \(\begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}\)

Hence \(a = a+c\) and \(a+b = b+d\)  
i.e. \(c = 0\) and \(d = a\)

For considering \(AQ = QA\) with general A  
M1 For correct simplified equation  
A1 For equating corresponding entries  
A1 For complete proof

(ii) To be non-singular, \(a \neq 0\)

For stating that \(a\) is non-zero

(iii) Identity is \(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\) as usual, since this is in \(S\)

Inverse of \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) is \(\begin{pmatrix} 1/a & -b/a^2 \\ 0 & 1/a \end{pmatrix}\), as \(a \neq 0\)

For justifying the identity correctly  
B1 For statement of correct inverse  
B1 For justification via non-zero \(a\)  
M1 For considering a general product  
A1 For complete proof

### 7 (i) \(z^n = \cos n\theta + i\sin n\theta\)  
\(z^{-n} = \cos n\theta - i\sin n\theta\), hence \(z^n + z^{-n} = 2\cos n\theta\)

For applying de Moivre’s theorem

(ii) \(2^6 \cos^6 \theta = (z + z^{-1})^6\)

\[\begin{align*}
&= (z^6 + z^{-6}) + 6(z^4 + z^{-4}) + 15(z^2 + z^{-2}) + 20 \\
&= 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20
\end{align*}\)

Hence \(\cos^6 \theta = \frac{1}{32}(3\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)\)

For correct substitution of multiple angles  
A1 For correct answer  
M1 For expanding and grouping terms

Integral is \(\frac{1}{32} \int_0^{10} (\sin 6\theta + 2\sin 4\theta + \frac{15}{2}\sin 2\theta + 100 \theta^\frac{1}{n}) d\theta\)

For integrating multiple angle expression

\[\begin{align*}
&= \frac{1}{32}(0 + \frac{3}{2}(-\frac{1}{2}\sqrt{3}) + \frac{15}{2}(\frac{1}{2}\sqrt{3}) + 10\times\frac{1}{n} \pi) \\
&= \frac{1}{32}(3\sqrt{3} + \frac{15}{2} \pi)
\end{align*}\)

For correct terms  
A1 For use of limits  
A1 For correct answer

For complete proof
8 (i) \( y = kx^2 e^{-2x} \Rightarrow y' = 2kxe^{-2x} - 2kx^2 e^{-2x} \) and
\( y'' = 2ke^{-2x} - 8kxe^{-2x} + 4kx^2 e^{-2x} \)
\( (2k - 8kx + 4kx^2 + 8kx - 8kx^2 + 4kx^2)e^{-2x} = 2e^{-2x} \)
Hence \( k = 1 \)

| M1 | For differentiation at least once |
| A1 | For both \( y' \) and \( y'' \) correct |
| M1 | For substituting completely in D.E. |
| A1 | 4 For correct value of \( k \) |

(ii) Auxiliary equation is \( m^2 + 4m + 4 = 0 \Rightarrow m = -2 \)

Hence C.F. is \( (A + Bx)e^{-2x} \)

G.S. is \( y = (A + Bx)e^{-2x} + x^2 e^{-2x} \)
\( x = 0, y = 1 \Rightarrow 1 = A \)
\( y' = B e^{-2x} - 2(A + Bx)e^{-2x} + 2xe^{-2x} - 2x^2 e^{-2x} \)
\( x = 0, y' = 0 \Rightarrow 0 = B - 2A \Rightarrow B = 2 \)

Hence solution is \( y = (1 + x)^2 e^{-2x} \)

| B1 | For correct repeated root |
| B1 | For correct form of C.F. |
| B1 | For sum of C.F. and P.I. |
| M1 | For using given values of \( x \) and \( y \) in G.S. |
| M1 | For differentiating the G.S. |
| M1 | For using given values of \( x \) and \( y' \) in G.S. |
| A1 | 7 For correct answer |

(iii) \( \frac{d^2 y}{dx^2} = 2 - 4 = -2 \) when \( x = 0 \)

Hence \( (0, 1) \) is a maximum point

\( \frac{dy}{dx} = 2(1+x)^2 e^{-2x} - 2(1+x)^2 e^{-2x} = -2x(1+x)e^{-2x} \),
so there are no turning points for \( x > 0 \)

Hence \( 0 < y \leq 1 \), since \( y \rightarrow 0 \) as \( x \rightarrow \infty \)

| B1 | For correct value \(-2\) |
| B1 | For statement of maximum at \( x = 0 \) |
| M1 | For investigation of turning points, or equiv |
| A1 | 4 For complete proof of given result |
INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer all the questions.
- Give non-exact numerical answers correct to 3 significant figures, unless a different degree of accuracy is specified in the question or is clearly appropriate.
- Where a numerical value for the acceleration due to gravity is needed, use 9.8 m s\(^{-2}\).
- You are permitted to use a graphic calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.
- You are reminded of the need for clear presentation in your answers.
An engine pulls a truck of mass 6000 kg along a straight horizontal track, exerting a constant horizontal force of magnitude \( E \) newtons on the truck (see diagram). The resistance to motion of the truck has magnitude 400 N, and the acceleration of the truck is \( 0.2 \text{ m s}^{-2} \). Find the value of \( E \). \[4\]

Forces of magnitudes 8 N and 5 N act on a particle. The angle between the directions of the two forces is \( 30^\circ \), as shown in Fig. 1. The resultant of the two forces has magnitude \( R \) N and acts at an angle \( \theta^\circ \) to the force of magnitude 8 N, as shown in Fig. 2. Find \( R \) and \( \theta \). \[7\]

A particle is projected vertically upwards, from the ground, with a speed of 128 m s\(^{-1}\). Ignoring air resistance, find

(i) the maximum height reached by the particle, \[2\]
(ii) the speed of the particle when it is 30 m above the ground, \[3\]
(iii) the time taken for the particle to fall from its highest point to a height of 30 m, \[3\]
(iv) the length of time for which the particle is more than 30 m above the ground. \[2\]
A woman runs from $A$ to $B$, then from $B$ to $A$ and then from $A$ to $B$ again, on a straight track, taking 90 s. The woman runs at a constant speed throughout. Fig. 1 shows the $(t, v)$ graph for the woman.

(i) Find the total distance run by the woman. [3]

(ii) Find the distance of the woman from $A$ when $t = 50$ and when $t = 80$. [3]

A child also starts to move, from $A$, along $AB$. The child walks at a constant speed for the first 50 s and then at an increasing speed for the next 40 s. Fig. 2 shows the $(t, v)$ graph for the child; it consists of two straight line segments.

(iii) At time $t = 50$, the woman and the child pass each other, moving in opposite directions. Find the speed of the child during the first 50 s. [3]

(iv) At time $t = 80$, the woman overtakes the child. Find the speed of the child at this instant. [3]

A particle $P$ moves in a straight line so that, at time $t$ seconds after leaving a fixed point $O$, its acceleration is $-\frac{1}{10}t$ m s$^{-2}$. At time $t = 0$, the velocity of $P$ is $V$ m s$^{-1}$.

(i) Find, by integration, an expression in terms of $t$ and $V$ for the velocity of $P$. [4]

(ii) Find the value of $V$, given that $P$ is instantaneously at rest when $t = 10$. [2]

(iii) Find the displacement of $P$ from $O$ when $t = 10$. [4]

(iv) Find the speed with which the particle returns to $O$. [3]
Three uniform spheres $A$, $B$ and $C$ have masses 0.3 kg, 0.4 kg and $m$ kg respectively. The spheres lie in a smooth horizontal groove with $B$ between $A$ and $C$. Sphere $B$ is at rest and spheres $A$ and $C$ are each moving with speed 3.2 m s$^{-1}$ towards $B$ (see diagram). Air resistance may be ignored.

(i) $A$ collides with $B$. After this collision $A$ continues to move in the same direction as before, but with speed 0.8 m s$^{-1}$. Find the speed with which $B$ starts to move. [4]

(ii) $B$ and $C$ then collide, after which they both move towards $A$, with speeds of 3.1 m s$^{-1}$ and 0.4 m s$^{-1}$ respectively. Find the value of $m$. [4]

(iii) The next collision is between $A$ and $B$. Explain briefly how you can tell that, after this collision, $A$ and $B$ cannot both be moving towards $C$. [1]

(iv) When the spheres have finished colliding, which direction is $A$ moving in? What can you say about its speed? Justify your answers. [4]

A sledge of mass 25 kg is on a plane inclined at $30^\circ$ to the horizontal. The coefficient of friction between the sledge and the plane is 0.2.

(i)

The sledge is pulled up the plane, with constant acceleration, by means of a light cable which is parallel to a line of greatest slope (see Fig. 1). The sledge starts from rest and acquires a speed of 0.8 m s$^{-1}$ after being pulled for 10 s. Ignoring air resistance, find the tension in the cable. [6]

(ii)

On a subsequent occasion the cable is not in use and two people of total mass 150 kg are seated in the sledge. The sledge is held at rest by a horizontal force of magnitude $P$ newtons, as shown in Fig. 2. Find the least value of $P$ which will prevent the sledge from sliding down the plane. [7]
1  \( E - 400 = 6000 \times 2 \)  
   \[ E = 1600 \]  
   \( \text{For resultant force } E - 400 \text{ stated or implied} \)  
   \( \text{M1} \)  
   \( \text{For use of Newton II for the truck} \)  
   \( \text{A1} \)  
   \( \text{For the correct equation} \)  
   \( \text{A1} \)  
   \( \text{For correct answer 1600} \)  

2  **EITHER:**  
   \( R \cos \theta = 8 + 5 \cos 30^\circ \)  
   \( R \sin \theta = 5 \sin 30^\circ \)  
   Hence  \( R^2 = (12.33\ldots)^2 + 2.5^2 \)  
   \( R = 12.6 \)  
   \[ \tan \theta = \frac{2.5}{12.33\ldots} \]  
   \( \theta = 11.5^\circ \)  
   \( \text{E} \)  
   \( \text{B1} \)  
   \( \text{For attempt at resolving } \parallel \text{ or } \perp \text{ to 8 N force} \)  
   \( \text{A1} \)  
   \( \text{For one completely correct equation} \)  
   \( \text{A1} \)  
   \( \text{For a second correct equation} \)  
   \( \text{A1} \)  
   \( \text{For correct method for either unknown} \)  
   \( \text{A1} \)  
   \( \text{For correct value} \)  

   **OR:**  
   Triangle of forces has 5, 8, \( R \) and 150°  
   \( R^2 = 8^2 + 5^2 - 2 \times 5 \times 8 \times \cos 150^\circ \)  
   Hence  \( R = 12.6 \)  
   \[ \sin \theta = \frac{5 \sin 150^\circ}{12.58\ldots} = 0.1987\ldots \]  
   Hence  \( \theta = 11.5^\circ \)  
   \( \text{E} \)  
   \( \text{B1} \)  
   \( \text{For considering any triangle with 5, 8, } R \)  
   \( \text{A1} \)  
   \( \text{For correct triangle drawn or used} \)  
   \( \text{A1} \)  
   \( \text{For use of cosine formula attempted} \)  
   \( \text{A1} \)  
   \( \text{For correct expression for } R^2 \)  
   \( \text{A1} \)  
   \( \text{For correct method for either unknown} \)  
   \( \text{A1} \)  
   \( \text{For correct value} \)  

3  (i)  \( 0 = 28t^2 - 2 \times 9.8 \times h \)  
   Hence maximum height is 40 m  
   \( \text{M1} \)  
   \( \text{For use of const acc formula(s) to find } h \)  
   \( \text{A1} \)  
   \( \text{For correct value 40} \)  

   (ii)  \( v^2 = 28^2 - 2 \times 9.8 \times 30 \)  
   Hence speed is 14 m s\(^{-1}\)  
   \( \text{M1} \)  
   \( \text{For use of const acc formula(s) to find } v \)  
   \( \text{A1} \)  
   \( \text{For correct equation in } v \)  
   \( \text{A1} \)  
   \( \text{For correct value 14} \)  

   (iii)  \( 10 = \frac{1}{2} \times 9.8t^2 \)  
   Hence time is 1.43 s  
   \( \text{A1} \)  
   \( \text{For use of const acc formula(s) to find } t \)  
   \( \text{A1} \)  
   \( \text{For correct equation in } t \)  
   \( \text{A1} \)  
   \( \text{For correct value 1.43 or equivalent} \)  

   (iv)  Length of time is  \( 2 \times \frac{10}{7} = \frac{20}{7} \) s  
   \( \text{A1} \)  
   \( \text{For doubling, or equiv longer method} \)  
   \( \text{A1} \)  
   \( \text{For correct value, i.e. double their (iii)} \)  
   \( \text{B1} \)  
   \( \text{For suitable use of } \frac{1}{2} (u + v)t \text{ or equiv} \)  

4  (i)  Total distance is \( 3 \times 30 + 3 \times 30 + 3 \times 30 = 270 \) m  
   \( \text{M1} \)  
   \( \text{For any calculation of a rectangular area} \)  
   \( \text{A1} \)  
   \( \text{For addition of three positive areas} \)  
   \( \text{A1} \)  
   \( \text{For correct value 270} \)  

   (ii)  Distance at \( t = 50 \) is 90 – 60 = 30 m  
   \( \text{M1} \)  
   \( \text{For correct use of signed areas} \)  
   \( \text{A1} \)  
   \( \text{For correct value 30} \)  
   \( \text{A1} \)  
   \( \text{For correct value 60} \)  

   (iii)  Child’s speed is \( \frac{30}{50} = 0.6 \) m s\(^{-1}\)  
   \( \text{B1} \)  
   \( \text{For distance 30 m} \)  
   \( \text{M1} \)  
   \( \text{For dividing by 50} \)  
   \( \text{A1} \)  
   \( \text{For correct value 0.6} \)  

   (iv)  Child walks 60 – 30 = 30 m in next 30 s  
   Hence \( 30 = \frac{1}{2} (0.6 + v) \times 30 \)  
   i.e. child’s speed is 1.4 m s\(^{-1}\)  
   \( \text{B1} \)  
   \( \text{For child’s distance gone from } t = 50 \text{ to 80} \)  
   \( \text{M1} \)  
   \( \text{For suitable use of } s = \frac{1}{2} (u + v)t \text{ or equiv} \)  
   \( \text{A1} \)  
   \( \text{For correct value 1.4} \)
5 (i) \[ v = \int -\frac{1}{10} t \, dt = -\frac{1}{20} t^2 + c \]
\[ V = 0 + c \]
\[ \text{Hence } v = V - \frac{1}{20} t^2 \]

M1 For integrating the acceleration formula
A1 For \( v = -\frac{1}{20} t^2 \), with or without \( c \)
M1 For using \( v = V \) when \( t = 0 \) to find \( c \)
A1 4 For correct equation for \( v \) in terms of \( t \) and \( V \)

(ii) \[ 0 = V - \frac{10^2}{20} \Rightarrow V = 5 \]

M1 For use of given values to find \( V \)
A1 2 For correct value 5

(iii) \[ s = \int \left( 5 - \frac{1}{20} t^2 \right) \, dt = 5t - \frac{1}{60} t^3 + k \]
\[ \text{Hence displacement is } 50 - \frac{1000}{60} = 33 \frac{1}{3} \text{ m} \]

M1 For any attempt to integrate velocity
A1 √ For correct integration (ignoring \( k \))
M1 For evaluation of \( s \) when \( t = 10 \)
A1 √ 4 For correct value \( 33 \frac{1}{3} \); allow omission of \( k \)

(iv) Returns to \( O \) when \( 0 = -\frac{1}{60} t^2 + 5 \Rightarrow t^2 = 300 \)
When \( t^2 = 300, \ v = -\frac{1}{20} \times 300 + 5 \)
i.e. speed is 10 m s\(^{-1}\)

M1 For attempting non-zero root of \( s = 0 \)
M1 For consequent evaluation of \( v \)
A1 3 For correct value 3 (allow negative here)

6 (i) \[ 0.3 \times 3.2 = 0.3 \times 0.8 + 0.4 \times b \]
\[ \text{Hence } b = 1.8 \text{ so } B \text{’s speed is } 1.8 \text{ m s}^{-1} \]

M1 For using conservation of momentum
A1 For correct LHS
A1 For correct RHS
A1 4 For correct value 1.8 correctly obtained

(ii) \[ 0.4 \times 1.8 - 3.2m = -0.4 \times 3.1 - 0.4m \]
\[ \text{Hence } m = 0.7 \]

M1 For momentum equn with at least one relevant negative sign
A1 For correct LHS
A1 For correct RHS
A1 4 For correct value 0.4 correctly obtained

(iii) \[ 0.4 \times 3.1 > 0.3 \times 0.8, \text{ so net momentum of } A \text{ and } B \]
\[ \text{is towards the left and therefore they can’t both move towards the right after the impact} \]

B1 1 For correctly explained application of momentum conservation.

(iv) Total momentum of all three particles is leftwards
\[ \text{Hence } A \text{ ends up moving left, as if it moves right after all collisions so do } B \text{ and } C \]
\[ \text{Total momentum left is at most } 1.4a \]
\[ \text{Hence } 1.4a \geq 0.7 \times 3.2 - 0.3 \times 3.2, \text{ so the speed of } A \text{ is at least } 0.914 \text{ m s}^{-1} \]

M1 For reasoning based on the total momentum
A1 For correct conclusion regarding direction
M1 For use of the idea that \( a \geq b \geq c \)
A1 4 For correct conclusion

13
7

(i) Acceleration is \( \frac{0.8}{10} = 0.08 \) m s\(^{-2}\)

\[ R = 25g \cos 30^\circ \]
\[ T = 25g \sin 30^\circ - 0.2 \times 25g \cos 30^\circ = 25 \times 0.08 \]

Hence the tension is 167 N

(ii) \( R' = P \sin 30^\circ + 175g \cos 30^\circ \)

\[ P \cos 30^\circ + 0.2R' = 175g \sin 30^\circ \]
\[ P(\cos 30^\circ + 0.2 \sin 30^\circ) = 175g(\sin 30^\circ - 0.2 \cos 30^\circ) \]

Hence \( P = \frac{175g(\sin 30^\circ - 0.2 \cos 30^\circ)}{\cos 30^\circ + 0.2 \sin 30^\circ} = 580 \)

For 0.8\(\times\)10 stated or implied

For correct resolving \( \perp \) plane

For attempting Newton II \( \parallel \) plane

For upwards force \( T = 25g \sin 30^\circ - F \)

For \( F = 0.2 \times 25g \cos 30^\circ \)

For correct value 167

For resolving \( \perp \) plane, with 3 forces

For correct equation

For resolving \( \parallel \) plane, with 3 forces

For correct equation

For attempting elimination of \( R' \)

For solving a relevant equation for \( P \)

For correct value 580
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1

A barge $B$ is pulled along a canal by a horse $H$, which is on the tow-path. The barge and the horse move in parallel straight lines and the tow-rope makes a constant angle of $15^\circ$ with the direction of motion (see diagram). The tow-rope remains taut and horizontal, and has a constant tension of 500 N.

(i) Find the work done on the barge by the tow-rope, as the barge travels a distance of 400 m. [3]

The barge moves at a constant speed and takes 10 minutes to travel the 400 m.

(ii) Find the power applied to the barge. [2]

2

A uniform circular cylinder, of radius 6 cm and height 15 cm, is in equilibrium on a fixed inclined plane with one of its ends in contact with the plane.

(i) Given that the cylinder is on the point of toppling, find the angle the plane makes with the horizontal. [3]

The cylinder is now placed on a horizontal board with one of its ends in contact with the board. The board is then tilted so that the angle it makes with the horizontal gradually increases.

(ii) Given that the coefficient of friction between the cylinder and the board is $\frac{3}{4}$, determine whether or not the cylinder will slide before it topples, justifying your answer. [4]

3

A uniform lamina $ABCD$ has the shape of a square of side $a$ adjoining a right-angled isosceles triangle whose equal sides are also of length $a$. The weight of the lamina is $W$. The lamina rests, in a vertical plane, on smooth supports at $A$ and $D$, with $AD$ horizontal (see diagram).

(i) Show that the centre of mass of the lamina is at a horizontal distance of $\frac{11}{7}a$ from $A$. [4]

(ii) Find, in terms of $W$, the magnitudes of the forces on the supports at $A$ and $D$. [4]
A rigid body $ABC$ consists of two uniform rods $AB$ and $BC$, rigidly joined at $B$. The lengths of $AB$ and $BC$ are 13 cm and 20 cm respectively, and their weights are 13 N and 20 N respectively. The distance of $B$ from $AC$ is 12 cm. The body hangs in equilibrium, with $AC$ horizontal, from two vertical strings attached at $A$ and $C$. Find the tension in each string. [8]

A cyclist and his machine have a combined mass of 80 kg. The cyclist ascends a straight hill $AB$ of constant slope, starting from rest at $A$ and reaching a speed of 5 m s$^{-1}$ at $B$. The level of $B$ is 4 m above the level of $A$.

(i) Find the gain in kinetic energy and the gain in gravitational potential energy of the cyclist and his machine. [3]

During the ascent the resistance to motion is constant and has magnitude 70 N.

(ii) Given that the work done by the cyclist in ascending the hill is 8000 J, find the distance $AB$. [3]

At $B$ the cyclist is working at 720 watts and starts to move in a straight line along horizontal ground. The resistance to motion has the same magnitude of 70 N as before.

(iii) Find the acceleration with which the cyclist starts to move horizontally. [4]

An athlete ‘puts the shot’ with an initial speed of 19 m s$^{-1}$ at an angle of 11° above the horizontal. At the instant of release the shot is 1.53 m above the horizontal ground. By treating the shot as a particle and ignoring air resistance, find

(i) the maximum height, above the ground, reached by the shot, [4]

(ii) the horizontal distance the shot has travelled when it hits the ground. [6]
A ball of mass 0.08 kg is attached by two strings to a fixed vertical post. The strings have lengths 2.5 m and 2.4 m, as shown in the diagram. The ball moves in a horizontal circle, of radius 2.4 m, with constant speed \( v \text{ m s}^{-1} \). Each string is taut and the lower string is horizontal. The modelling assumptions made are that both strings are light and inextensible, and that there is no air resistance.

(i) Find the tension in each string when \( v = 10.5 \). \[7\]

(ii) Find the least value of \( v \) for which the lower string is taut. \[4\]

Two uniform smooth spheres, \( A \) and \( B \), have the same radius. The mass of \( A \) is 0.24 kg and the mass of \( B \) is \( m \) kg. Sphere \( A \) is travelling in a straight line on a horizontal table, with speed \( 8 \text{ m s}^{-1} \), when it collides directly with sphere \( B \), which is at rest. As a result of the collision, sphere \( A \) continues in the same direction with a speed of \( 6 \text{ m s}^{-1} \).

(i) Find the magnitude of the impulse exerted by \( A \) on \( B \). \[3\]

(ii) Show that \( m \leq 0.08 \). \[3\]

It is given that \( m = 0.06 \).

(iii) Find the coefficient of restitution between \( A \) and \( B \). \[3\]

On another occasion \( A \) and \( B \) are travelling towards each other, each with speed \( 4 \text{ m s}^{-1} \), when they collide directly.

(iv) Find the speeds of \( A \) and \( B \) immediately after the collision. \[4\]
1  (i) Work done is $500\cos15^\circ \times 400 = 193\,000\,J$  
M1  For attempt to use $\text{Force} \times \text{distance}$  
A1  For correct unsimplified product  
A1  For correct answer $193\,000$

(ii) Power applied is $\frac{193\,185}{600} = 322\,W$  
M1  For relevant use of $\frac{\text{work}}{\text{time}}$ or $\text{force} \times \text{velocity}$  
A1  For correct answer $322$

2  (i) CM is vertically above lowest point of base  
Hence $\tan \alpha = \frac{6}{7.5} \Rightarrow \alpha = 38.7^\circ$  
B1  For stating or implying correct geometry  
M1  For appropriate trig calculation  
A1  For correct answer $38.7$

(ii) Cylinder slides when $\tan \theta = \frac{3}{4}$  
But $\frac{3}{4} < 0.8$, so $\theta < \alpha$  
B1  For stating or implying limiting friction case  
M1  For comparing $\tan \alpha$ to $\tan \theta$, or equivalent  
A1  For correct comparison of the angles  
A1  For correct conclusion of sliding first

3  (i) CG of triangle is $\frac{1}{3}a$ horizontally from $A$  
Moments: $\frac{1}{2}W \times \frac{1}{3}a + \frac{1}{2}W \times \frac{1}{2}a = W \times \bar{x}$  
B1  For equating moments about $A$, or equivalent  
M1  For a correct unsimplified equation  
A1  Given answer correctly shown

(ii) $R_A \times 2a = W \times \frac{1}{3}a \Rightarrow R_A = \frac{2}{18}W$  
$R_A + R_D = W \Rightarrow R_D = \frac{11}{18}W$  
M1  For one moments equation  
A1  For one correct answer  
M1  For resolving, or a second moments equation  
A1  For a second correct answer

4  Horiz distances of $B$ from $A$ and $C$ are 5 cm and 16 cm  
$21T_A = 13 \times 18.5 + 20 \times 8$  
$T_A + T_C = 33$  
Hence $T_A = 19.1\,N$ and $T_C = 13.9\,N$  
M1  For appropriate use of Pythagoras  
A1  For both distances correct  
M1  For any moments equation for the system  
A1  For any one relevant term correct  
A1  For a completely correct equation  
M1  For resolving, or using another moments eqn  
A1  For correct answer $19.1$  
A1  For correct answer $13.9$

5  (i) Gain in KE is $\frac{1}{2} \times 80 \times 5^2 = 1000\,J$  
Gain in PE is $80 \times 9.8 \times 4 = 3136\,J$  
M1  For use of formula $\frac{1}{2}m v^2$  
M1  For use of formula $mgh$  
A1  For both answers $1000$ and $3136$ correct

(ii) $8000 = 1000 + 3136 + 70d$  
Hence distance $AB$ is $55.2\,m$  
M1  For equating work done to energy change  
M1  For relevant use of $\text{force} \times \text{distance}$  
A1  For correct answer $55.2$

(iii) $\frac{720}{5} - 70 = 80a$  
Hence acceleration is $0.925\,m\,s^{-2}$  
B1  For driving force $\frac{720}{5}$  
M1  For use of Newton II with 3-term equation  
A1  For a completely correct equation  
A1  For correct answer $0.925$
### 6

(i) \[ 0 = (19\sin11^\circ)^2 - 2gh \]

Hence max height is \( \frac{(19\sin11^\circ)^2}{19.6} + 1.53 = 2.20 \) m

- M1 For use of relevant const acc equation for \( h \)
- B1 For correct vertical component \( 19\sin11^\circ \)
- A1 For correct expression for \( h (= 0.67) \)
- A1 For correct answer 2.20

(ii) EITHER: Time to top point is \( \frac{19\sin11^\circ}{g} = 0.3699 \)

Time to fall is \( \sqrt{\frac{2 \times 2.20}{9.8}} = 0.6701 \)

- M1 For use of relevant const acc eqn for \( t_{up} \)
- M1 For use of relevant const acc eqn for \( t_{down} \)
- A1 For a correct expression for \( t_{down} \)
- A1 For correct value (or expression)
- M1 For any use of \( x = (19\cos11^\circ)t \)
- A1 For correct answer 19.4

**OR:**

\[ -1.53 = x\tan11^\circ - \frac{gx^2}{2 \times (19\cos11^\circ)^2} \]

Hence \( x = 19.4 \)

- M1 For relevant use of trajectory equation
- B1 For \( y = -1.53 \) correctly substituted
- A1 For completely correct equation for \( x \)
- M1 For attempt to solve relevant quadratic
- A2 6 For correct answer 19.4

### 7

(i) \( T_1 \times \frac{2}{25} = 0.08g \)

Hence tension in upper string is 2.8 N

\[ T_1 \times \frac{24}{25} + T_2 = 0.08 \times \frac{10.5^2}{2.4} \]

- M1 For resolving vertically
- B1 For \( \frac{2}{25} \) or \( \sin16.3^\circ \) or equivalent
- A1 For correct value 2.8

Hence tension in horizontal string is 0.987 N

- M1 For correct use of Newton II horizontally
- B1 For any use of \( \frac{10.5^2}{2.4} \), or equivalent
- A1 For correct horizontal equation
- A1 7 For correct value 0.987

(ii) \( 2.8 \times \frac{2.4}{2.5} = 0.08 \times \frac{v^2}{2.4} \)

Hence \( v = 8.98 \)

- M1 For new horizontal equation with \( T_2 = 0 \)
- A1 \( \sqrt{\} \) For correct equation for \( v \)
- M1 For solving for \( v \) correctly
- A1 4 For correct value 8.98
8  

(i) Change of momentum of $A$ is $0.24 \times 2$

Hence magnitude of impulse is 0.48 N s

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<tr>
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<tbody>
<tr>
<td>M1</td>
<td>For considering momentum of $A$</td>
</tr>
<tr>
<td>A1</td>
<td>For correct expression for change in mom</td>
</tr>
<tr>
<td>A1 3</td>
<td>For correct answer 0.48</td>
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(ii) $mv_B = 0.48$

$v_B \geq 6$

Hence $m = \frac{0.48}{6} = 0.08$

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<tr>
<td>M1</td>
<td>For considering momentum of $B$</td>
</tr>
<tr>
<td>M1</td>
<td>For using the inequality $v_B \geq v_A$</td>
</tr>
<tr>
<td>A1 3</td>
<td>For showing given answer correctly</td>
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</table>

(iii) $m = 0.06 \Rightarrow v_B = 8$

Hence $8 - 6 = e(8 - 0)$

i.e. $e = \frac{1}{4}$

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<tr>
<td>B1</td>
<td>For correct speed of $B$</td>
</tr>
<tr>
<td>M1</td>
<td>For correct use of Newton’s law</td>
</tr>
<tr>
<td>A1 3</td>
<td>For correct answer $\frac{1}{4}$ or equivalent</td>
</tr>
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</table>

(iv) $0.24 \times 4 - 0.06 \times 4 = 0.24a + 0.06b$

$b - a = \frac{1}{4}(4 + 4)$

Hence speeds of $A$ and $B$ are 2 m s$^{-1}$ and 4 m s$^{-1}$

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<tr>
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<tbody>
<tr>
<td>B1</td>
<td>For a correct momentum equation</td>
</tr>
<tr>
<td>B1$\checkmark$</td>
<td>For a correct restitution equation</td>
</tr>
<tr>
<td>M1</td>
<td>For solution of relevant simultaneous equns</td>
</tr>
<tr>
<td>A1 4</td>
<td>For both answers correct</td>
</tr>
</tbody>
</table>

| Total | 13 |

4739 Specimen Paper
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1 A particle is moving with simple harmonic motion in a straight line. The period is 0.2 s and the amplitude of the motion is 0.3 m. Find the maximum speed and the maximum acceleration of the particle. [6]

2 A sphere $A$ of mass $m$, moving on a horizontal surface, collides with another sphere $B$ of mass $2m$, which is at rest on the surface. The spheres are smooth and uniform, and have equal radius. Immediately before the collision, $A$ has velocity $u$ at an angle $\theta^o$ to the line of centres of the spheres (see diagram). Immediately after the collision, the spheres move in directions that are perpendicular to each other.

(i) Find the coefficient of restitution between the spheres. [4]

(ii) Given that the spheres have equal speeds after the collision, find $\theta$. [3]

3 An aircraft of mass $80000 \text{ kg}$ travelling at $90 \text{ m s}^{-1}$ touches down on a straight horizontal runway. It is brought to rest by braking and resistive forces which together are modelled by a horizontal force of magnitude $(27000 + 50v^2) \text{ newtons}$, where $v \text{ m s}^{-1}$ is the speed of the aircraft. Find the distance travelled by the aircraft between touching down and coming to rest. [8]

4 For a bungee jump, a girl is joined to a fixed point $O$ of a bridge by an elastic rope of natural length 25 m and modulus of elasticity 1320 N. The girl starts from rest at $O$ and falls vertically. The lowest point reached by the girl is 60 m vertically below $O$. The girl is modelled as a particle, the rope is assumed to be light, and air resistance is neglected.

(i) Find the greatest tension in the rope during the girl’s jump. [2]

(ii) Use energy considerations to find

(a) the mass of the girl, [4]

(b) the speed of the girl when she has fallen half way to the lowest point. [3]
A particle $P$ of mass 0.3 kg is moving in a vertical circle. It is attached to the fixed point $O$ at the centre of the circle by a light inextensible string of length 1.5 m. When the string makes an angle of 40° with the downward vertical, the speed of $P$ is 6.5 m s$^{-1}$ (see diagram). Air resistance may be neglected.

(i) Find the radial and transverse components of the acceleration of $P$ at this instant. [2]

In the subsequent motion, with the string still taut and making an angle $\theta^\circ$ with the downward vertical, the speed of $P$ is $v$ m s$^{-1}$

(ii) Use conservation of energy to show that $v^2 = 19.7 + 29.4\cos\theta^\circ$. [4]

(iii) Find the tension in the string in terms of $\theta$. [4]

(iv) Find the value of $v$ at the instant when the string becomes slack. [3]

A step-ladder is modelled as two uniform rods $AB$ and $AC$, freely jointed at $A$. The rods are in equilibrium in a vertical plane with $B$ and $C$ in contact with a rough horizontal surface. The rods have equal lengths; $AB$ has weight 150 N and $AC$ has weight 270 N. The point $A$ is 2.5 m vertically above the surface, and $BC = 1.6$ m (see diagram).

(i) Find the horizontal and vertical components of the force acting on $AC$ at $A$. [8]

(ii) The coefficient of friction has the same value $\mu$ at $B$ and at $C$, and the step-ladder is on the point of slipping. Giving a reason, state whether the equilibrium is limiting at $B$ or at $C$, and find $\mu$. [6]
Two points $A$ and $B$ lie on a vertical line with $A$ at a distance 2.6 m above $B$. A particle $P$ of mass 10 kg is joined to $A$ by an elastic string and to $B$ by another elastic string (see diagram). Each string has natural length 0.8 m and modulus of elasticity 196 N. The strings are light and air resistance may be neglected.

(i) Verify that $P$ is in equilibrium when $P$ is vertically below $A$ and the length of the string $PA$ is 1.5 m. \[4\]

The particle is set in motion along the line $AB$ with both strings remaining taut. The displacement of $P$ below the equilibrium position is denoted by $x$ metres.

(ii) Show that the tension in the string $PA$ is $245(0.7 + x)$ newtons, and the tension in the string $PB$ is $245(0.3 - x)$ newtons. \[3\]

(iii) Show that the motion of $P$ is simple harmonic. \[3\]

(iv) Given that the amplitude of the motion is 0.25 m, find the proportion of time for which $P$ is above the mid-point of $AB$. \[5\]
1. \[ 0.2 = \frac{2\pi}{\omega} \Rightarrow \omega = 10\pi \]

Hence maximum speed is \[ 0.3 \times 10\pi = 3\pi = 9.42 \text{ m s}^{-1} \]

Maximum acc is \[ 0.3 \times (10\pi)^2 = 30\pi^2 = 296 \text{ m s}^{-2} \]

M1  For relevant use of \( \frac{2\pi}{\omega} \)

A1  For correct value 10\pi

M1  For relevant use of \( v = a\omega \)

A1\(^{\sqrt{}}\)  For correct value \( 3\pi \) or 9.42

M1  For relevant use of \( a\omega^2 \)

A1\(^{\sqrt{}}\)  For correct value 30\pi or 296

2. \( A \) and \( B \) move off \( \perp \) and \( \parallel \) resp. to line of centres

\[ 2mv_A = mu \cos \theta \]

\[ v_A = eu \cos \theta \]

Hence \( e = 0.5 \)

M1  For correct directions of motion after impact

A1  For correct momentum equation

A1  For correct restitution equation

4

So \( \theta = \tan^{-1} 0.5 = 26.6^\circ \)

M1  For forming the relevant equation for \( \theta \)

3

3. \[ 80000v \frac{dv}{dx} = -(27000 + 50v^2) \]

Hence \[ x = -\int \frac{1600v}{540 + v^2} dv \]

\[ = -800\ln(540 + v^2) + k \]

M1  For using Newton II to form a DE

A1  For correct equation including \( v \frac{dv}{dx} \)

M1  For separation of variables

M1  For logarithmic form of integral

\( v = 90 \) when \( x = 0 \Rightarrow k = 800\ln 8640 \)

M1  For use of initial condition to find \( k \)

M1  For evaluation of required distance

(\( \text{The previous two M marks can equivalently be earned by using definite integration} \)

So distance is 2220 m approximately

A1  For correct value 2220

8

4. \( i \) Greatest tension \[ \frac{1320 \times 35}{25} = 1848 \text{ N} \]

M1  For use of \( \frac{\lambda x^2}{l} \) at lowest point

A1  For correct answer 1848

\( ii \) \( a \)

\[ mg \times 60 = \frac{1320}{2 \times 25} \]

Hence the girl’s mass is 55 kg

M1  For use of correct EPE formula \( \frac{\lambda x^2}{2l} \)

A1  For correct unsimplified expression for EPE

M1  For use of equation involving EPE and GPE

A1  For correct answer 55

\( b \)

\[ 55g \times 30 = \frac{1}{2} \times 55v^2 + \frac{1320}{2 \times 25} \times (30 - 25)^3 \]

M1  For energy equation with KE, GPE and EPE

A1\(^{\sqrt{}}\)  For equation with all terms correct

A1  For correct answer 24.3

3
5 (i) Radial acc is \( \frac{6.5^2}{1.5} = 28.2 \text{ m s}^{-2} \)

Transverse acc is \( g \sin 40^\circ = 6.30 \text{ m s}^{-2} \)

(ii) \( \frac{1}{2} \times 0.3 \times (6.5^2 - v^2) = 0.3 \times 9.8 \times 1.5 (\cos 40^\circ - \cos \theta^\circ) \)

Hence \( 42.25 - v^2 = 29.4 (\cos 40^\circ - \cos \theta^\circ) \)

i.e. \( v^2 = 19.7 + 29.4 \cos \theta^\circ \)

(ii) \( T - 0.3g \cos \theta^\circ = 0.3 \times \frac{v^2}{1.5} \)

Hence \( T = 2.94 \cos \theta^\circ + 0.2 (19.7 + 29.4 \cos \theta^\circ) \)

\( = 3.95 + 8.82 \cos \theta^\circ \)

(iv) \( T = 0 \) when \( 3.95 + 8.82 \cos \theta^\circ = 0 \)

Hence \( v^2 = 19.7 + 29.4 \times \left( \frac{3.95}{8.82} \right) \Rightarrow v = 2.56 \)

<table>
<thead>
<tr>
<th>5</th>
<th>8</th>
<th>13</th>
</tr>
</thead>
</table>
| 6 (i) Mom @ B for BAC: \( V_C \times 1.6 = 150 \times 0.4 + 270 \times 1.2 \) Hence \( V_C = 240 \) | M1 | For suitable moments equation for BAC
Mom @ C for AC: \( V_A \times 0.8 + H_A \times 2.5 = 270 \times 0.4 \) Hence \( V_A = 270 - 240 = 30 \text{ N (upwards) and } 2.5H_A = 108 - 0.8 \times 30 \Rightarrow H_A = 33.6 \text{ N (right)} \) | A1 | For a moments equation for one rod with all required forces included
Res ‡ for AC: \( V_A + V_C = 270 \) Hence \( V_A = 270 - 240 = 30 \text{ N (upwards)} and 2.5H_A = 108 - 0.8 \times 30 \Rightarrow H_A = 33.6 \text{ N (right)} \) | M1 | For another equation leading to \( V_A \)
| (ii) \( V_B = 270 + 150 - V_C = 180 \) | M1 | For finding all of \( V_B, H_B \) and \( H_C \)
\( H_B = H_C = H_A = 33.6 \) \( \frac{H_B}{V_B} = 0.187, \frac{H_C}{V_C} = 0.14 \) Hence friction is limiting at B Value of \( \mu \) is 0.187 | A1\(^\swarrow\) | For considering ratios at B and C, or equiv
| | | For identifying point with larger ratio
| | | For identifying the larger ratio as \( \mu \) | A1\(^\swarrow\) | 14 |
### 7

#### (i)  
\[ T_{AP} = \frac{196}{0.8} \times (1.5 - 0.8) = 171.5 \]  
\[ T_{BP} = \frac{196}{0.8} \times (2.6 - 1.5 - 0.8) = 73.5 \]  
\[ T_{AP} - T_{BP} = 98 = 10g, \text{ hence equilibrium} \]

- **M1** For using Hook’s law to find either tension
- **A1** For both tensions correct
- **M1** For considering \( T_{AP} = mg + T_{BP} \), or equiv
- **A1** For showing given result correctly

#### (ii)  
Extension of \( PA \) is \( 1.5 + x - 0.8 = 0.7 + x \)

- **M1** For finding either extension in terms of \( x \)
- **A1** For showing one given answer correctly
- **A1** For showing the other given answer correctly

\[ T_{AP} = \frac{196}{0.8}(0.7 + x) = 245(0.7 + x) \]  
\[ T_{BP} = \frac{196}{0.8}(1.1 - x - 0.8) = 245(0.3 - x) \]

#### (iii)  
\[ 245(0.3 - x) + 10g = 245(0.7 + x) = 10\ddot{x} \]

- **M1** For use of Newton II, at a general position
- **A1** For a correct equation
- **A1** For showing the given result correctly

\[ \ddot{x} = -49x, \text{ so the motion is SHM} \]

#### (iv)  
\[ 0.2 = 0.25 \cos(7t) \]

- **M1** For use of \( \pm 0.2 \) in SHM equation involving \( t \)
- **A1** For a correct equation for a relevant time
- **A1** For correct value for a relevant time

\[ \text{Hence half of time above mid-pt is } t = 0.0919... \]  
\[ \text{Proportion is } \frac{t}{\pi/\omega} = 0.205 \]

- **M1** For relating \( t \) to period of oscillation
- **A1** For correct proportion 0.205

**Total Marks:** 15
INSTRUCTIONS TO CANDIDATES

• Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
• Answer all the questions.
• Give non-exact numerical answers correct to 3 significant figures, unless a different degree of accuracy is specified in the question or is clearly appropriate.
• Where a numerical value for the acceleration due to gravity is needed, use $9.8 \text{ m s}^{-2}$.
• You are permitted to use a graphic calculator in this paper.

INFORMATION FOR CANDIDATES

• The number of marks is given in brackets [ ] at the end of each question or part question.
• The total number of marks for this paper is 72.
• Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.
• You are reminded of the need for clear presentation in your answers.
1 A circular flywheel of radius 0.2 m is rotating freely about a fixed axis through its centre and perpendicular to its plane. The moment of inertia of the flywheel about the axis is 0.37 kg m$^2$. When the angular speed of the flywheel is 8 rad s$^{-1}$ a particle of mass 0.75 kg, initially at rest, sticks to a point on the circumference of the flywheel. Find

(i) the angular speed of the flywheel immediately after the particle has stuck to it, \[4\]
(ii) the loss of energy that results when the particle sticks to the flywheel. \[2\]

2 A uniform solid sphere, of mass 4 kg and radius 0.1 m, is rotating freely about a fixed axis with angular speed 120 rad s$^{-1}$. The axis is a diameter of the sphere. A couple, having constant moment 0.36 N m about the axis and acting in the direction of rotation, is then applied for 6 seconds. For this time interval, find

(i) the angular acceleration of the sphere, \[3\]
(ii) the angle through which the sphere turns, \[2\]
(iii) the work done by the couple. \[2\]

3 The region bounded by the $x$-axis, the $y$-axis, and the curve $y = 4 - x^2$ for $0 \leq x \leq 2$, is occupied by a uniform lamina of mass 35 kg. The unit of length is the metre. Show that the moment of inertia of the lamina about the $y$-axis is 28 kg m$^2$. \[8\]

4 A straight rod $AB$ of length $a$ has variable density, and at a distance $x$ from $A$ its mass per unit length is $k \left(1 + \frac{x^2}{a^2}\right)$, where $k$ is a constant.

(i) Find the distance of the centre of mass of the rod from $A$. \[6\]

You are given that the moment of inertia of the rod about a perpendicular axis through $A$ is $\frac{8}{15}ka^3$.

(ii) Show that the period of oscillation of the rod as a compound pendulum, when freely pivoted at the other end $B$, is $2\pi \sqrt{\frac{22a}{35g}}$. \[5\]

5 A uniform rod $AB$, of mass $m$ and length $2a$, is free to rotate in a vertical plane about a fixed horizontal axis through $A$. The rod is released from rest with $AB$ horizontal. Air resistance may be neglected. For the instant when the rod has rotated through an angle $\frac{1}{6}\pi$ ,

(i) show that the angular acceleration of the rod is $\frac{(3\sqrt{3})g}{8a}$, \[2\]
(ii) find the angular speed of the rod, \[3\]
(iii) show that the force acting on the rod at $A$ has magnitude $\frac{\sqrt{103}}{8}mg$. \[7\]
A cylinder with radius $a$ is fixed with its axis horizontal. A uniform rod, of mass $m$ and length $2b$, moves in a vertical plane perpendicular to the axis of the cylinder, maintaining contact with the cylinder and not slipping (see diagram). When the rod is horizontal, its mid-point $G$ is in contact with the cylinder. You are given that, when the rod makes an angle $\theta$ with the horizontal, the height of $G$ above the axis of the cylinder is $a(\theta \sin \theta + \cos \theta)$.

(i) By considering the potential energy of the rod, show that $\theta = 0$ is a position of stable equilibrium. [6]

(ii) You are also given that, when $\theta$ is small, the kinetic energy of the rod is approximately $\frac{1}{2} mb^2 \dot{\theta}^2$. Show that the approximate period of small oscillations about the position $\theta = 0$ is $\frac{2\pi b}{\sqrt{(3ga)}}$. [7]

An unidentified object $U$ is flying horizontally due east at a constant speed of $220 \text{ m s}^{-1}$. An aircraft is 15 000 m from $U$ and is at the same height as $U$. The bearing of $U$ from the aircraft is $310^\circ$.

(i) Assume that the aircraft flies in a straight line at a constant speed of $160 \text{ m s}^{-1}$.

(a) Find the bearings of the two possible directions in which the aircraft can fly to intercept $U$. [6]

(b) Given that the interception occurs in the shorter of the two possible times, find the time taken to make the interception. [5]

(ii) Assuming instead that the aircraft flies in a straight line at a constant speed of $130 \text{ m s}^{-1}$, show that the nearest the aircraft can come to $U$ is approximately 988 m. [4]
### 1 (i) MI with particle is $0.37 + 0.75 \times 0.2^2 = 0.4$

$0.4 \omega = 0.37 \times 8$

Hence angular speed is $7.4 \text{ rad s}^{-1}$

For $0.75 \times 0.2^2$

A1 For correct MI, stated or implied

M1 For relevant use of cons. of ang. mom.

A1 For correct value 7.4

(ii) K.E. loss $\frac{1}{2} \times 0.37 \times 8^2 - \frac{1}{2} \times 0.4 \times 7.4^2 = 0.888 \text{ J}$

For an correct relevant use of $\frac{1}{2} I \omega^2$

A1 For correct value for the KE loss

### 2 (i) $I = \frac{1}{2} \times 4 \times 0.1^2 = 0.016$

$0.36 = 0.016 \alpha$

Hence angular acceleration is $22.5 \text{ rad s}^{-2}$

For correct use of $\frac{1}{2} mr^2$

B1 For use of $C = I \alpha$ to find $\alpha$

A1 For correct value 22.5

(ii) $\theta = 20 \times 6 + \frac{1}{2} \times 22.5 \times 6^2$

Angle turned through is 525 radians

For use of $\theta = \alpha t + \frac{1}{2} \alpha^2 t^2$ to find $\theta$

A1 For correct answer 525

(iii) Work done $= 0.36 \times 525 = 189 \text{ J}$

For use of $C \theta$, or increase in $\frac{1}{2} I \omega^2$

A1 For correct answer 189

### 3 EITHER: Area is $\int_0^2 (4 - x^2) \, dx = \left[ 4x - \frac{1}{3} x^3 \right]_0^2 = \frac{16}{3}$

Hence $\frac{4}{3} \rho = 35 \Rightarrow \rho = \frac{105}{16}$

For evaluation of $\int_0^2 y \, dx$

M1 For correct value $\frac{16}{3}$

A1 For correct density

B1 For correct expression for $I$

$\int_0^2 \rho x^2 y \, dx = \frac{105}{16} \int_0^2 x^2 (4 - x^2) \, dx$

A1 For use of $\int x^2 y \, dx$

A1 For correct numerical expression $\frac{64}{3} \rho$

A1 For obtaining given answer 28 correctly

OR: Area is $\int_0^4 (4 - y^2)^{\frac{1}{2}} \, dy = \left[ -\frac{2}{3} (4 - y)^{\frac{3}{2}} \right]_0^4 = \frac{16}{3}$

Hence $\frac{4}{3} \rho = 35 \Rightarrow \rho = \frac{105}{16}$

For evaluation of $\int_0^4 x \, dy$

M1 For correct value $\frac{16}{3}$

A1 For correct density

B1 For correct expression for $I$

$\frac{1}{3} \rho \int_0^4 x^2 \, dy = \frac{35}{16} \int_0^4 (4 - y)^2 \, dy$

A1 For use of $\frac{1}{3} \int x^3 \, dy$

A1 For correct indefinite integral $-\frac{2}{3} (4 - y)^{\frac{3}{2}}$

A1 For correct numerical expression $\frac{64}{3} \rho$

A1 For obtaining given answer 28 correctly
3

### 4 (i) Moment @ A = \[ \int_0^a kx \left( 1 + \frac{x^2}{a^2} \right) dx = k \left[ \frac{x^2}{2} + \frac{x^4}{4a^2} \right]_0^a \]

\[ = \frac{1}{4} ka^2 \quad \text{M1} \quad \text{For attempted integration of } \rho x \text{ with limits} \]

Mass of rod is \[ \int_0^a k \left( 1 + \frac{x^2}{a^2} \right) dx = k \left[ x + \frac{x^3}{3a^2} \right]_0^a \]

\[ = \frac{1}{4} ka \quad \text{A1} \quad \text{For correct mass } \frac{1}{4} ka \]

Hence \[ \frac{1}{4} ka \bar{x} = \frac{1}{4} ka^2 \Rightarrow \bar{x} = \frac{9}{16} a \quad \text{M1} \quad \text{For moments equation for } \bar{x} \]

\[ \text{For attempted integration of } \rho \text{ with limits} \]

\[ = \frac{1}{4} ka \quad \text{A1} \quad \text{For correct value } \frac{1}{4} ka^3, \text{ or equivalent} \]

\[ \text{For showing given answer correctly} \]

---

### 4 (ii) \( I_G = I_A - m(\bar{x})^2 = \frac{1}{4} ka^3 - \frac{1}{4} ka \left( \frac{9}{16} a \right)^2 = \frac{107}{960} ka^3 \]

\( I_R = I_G + m(a - \bar{x})^2 = \frac{107}{960} ka^3 + \frac{1}{4} ka \left( \frac{7}{16} a \right)^2 = \frac{11}{30} ka^3 \quad \text{M1} \quad \text{For correct use of } || \text{ axes to find } I_R \]

Period is \[ 2\pi \sqrt{\frac{\frac{11}{30} ka^3}{\left( \frac{1}{4} ka \right) g \left( \frac{9}{16} a \right)}} = 2\pi \sqrt{\frac{22a}{35g}} \quad \text{M1} \quad \text{For correct use of } 2\pi \sqrt{\frac{I}{mg}} \]

\[ \text{For showing given answer correctly} \]

---

### 5 (i) \[ mga \cos \frac{1}{6} \pi = \frac{4}{3} ma^2 \alpha \]

Hence \[ \alpha = \frac{(3\sqrt{3}) g}{8a} \quad \text{A1} \quad \text{For obtaining given answer correctly} \]

---

### 5 (ii) \[ \frac{1}{2} \times \frac{4}{3} ma^2 \times \omega^2 = mga \sin \frac{1}{6} \pi \]

Hence \[ \omega = \sqrt{\frac{3g}{4a}} \quad \text{A1} \quad \text{For correct answer} \]

---

### 5 (iii) Res \( \parallel \) rod: \[ R - mg \sin \frac{1}{6} \pi = ma \omega^2 \]

Hence \[ R = \frac{1}{2} mg + \frac{3}{2} mg = \frac{5}{2} mg \quad \text{M1} \quad \text{For Newton II equation with 3 terms} \]

Res \( \perp \) rod: \[ mg \cos \frac{1}{6} \pi - S = ma \bar{\alpha} \]

Hence \[ S = \left( \frac{1}{2} \sqrt{3} \right) mg - \left( \frac{1}{2} \sqrt{3} \right) mg = \left( \frac{1}{2} \sqrt{3} \right) mg \]

Magnitude is \[ \sqrt{R^2 + S^2} = \frac{1}{2} mg \sqrt{(10^2 + 3)} \]

\[ = \sqrt{103} mg \quad \text{A1} \quad \text{For obtaining given answer correctly} \]
6 (i) \( V = mga(\theta \sin \theta + \cos \theta) \), so
\[
\frac{dV}{d\theta} = mga(\cos \theta \cos \theta + \sin \theta - \sin \theta) = mga\theta \cos \theta.
\]
Hence equilibrium at \( \theta = 0 \), since \( \frac{dV}{d\theta} = 0 \)
\[
\frac{d^2V}{d\theta^2} = mga(\cos \theta - \theta \sin \theta).
\]
When \( \theta = 0 \), \( \frac{d^2V}{d\theta^2} = mga > 0 \), so equim is stable

(ii) \( mga(\theta \sin \theta + \cos \theta) + \frac{1}{2} mb^2 \dot{\theta}^2 = K \)
Hence \( (mga\theta \cos \theta)\dot{\theta} + \frac{1}{2} mb^2 \dot{\theta}^2 = 0 \)
For small \( \theta \), \( mga\theta + \frac{1}{2} mb^2 \dot{\theta} = 0 \Rightarrow \ddot{\theta} = -\frac{3ga}{b^2} \theta 
\]
Motion is approximate SHM with period \( \frac{2\pi b}{\sqrt{(3ga)}} \)

7 (i) (a) \[
\begin{align*}
\sin \theta &= \sin 40^\circ \\
\frac{220}{220} &= \frac{160}{160} \\
Hence \theta &= 62.1^\circ, \phi = 117.9^\circ \\
Required bearings are 012.1^\circ and 067.9^\circ
\end{align*}
\]
(b) Shorter time occurs for \( \theta = 62.1^\circ \)
\[
\frac{v}{\sin 77.9^\circ} = \frac{160}{\sin 40^\circ} \Rightarrow v = 243.4
\]
Hence time is \( \frac{15000}{243.4} = 61.6 \text{ s} \)

(ii) For closest approach, \( \sin \alpha = \frac{130}{220} \Rightarrow \alpha = 36.2^\circ \)
Hence min distance is \( 15000\sin(40 - \alpha) = 988 \text{ m} \)
OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MATHEMATICS 4732

Probability and Statistics 1

Specimen Paper

Additional materials:
Answer booklet
Graph paper
List of Formulae (MF 1)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

• Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
• Answer all the questions.
• Give non-exact numerical answers correct to 3 significant figures, unless a different degree of accuracy is specified in the question or is clearly appropriate.
• You are permitted to use a graphic calculator in this paper.

INFORMATION FOR CANDIDATES

• The number of marks is given in brackets [ ] at the end of each question or part question.
• The total number of marks for this paper is 72.
• Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.
• You are reminded of the need for clear presentation in your answers.

This question paper consists of 4 printed pages.
1 Janet and John wanted to compare their daily journey times to work, so they each kept a record of their journey times for a few weeks.

(i) Janet’s daily journey times, \( x \) minutes, for a period of 25 days, were summarised by \( \Sigma x = 2120 \) and \( \Sigma x^2 = 180044 \). Calculate the mean and standard deviation of Janet’s journey times. \[3\]

(ii) John’s journey times had a mean of 79.7 minutes and a standard deviation of 6.22 minutes. Describe briefly, in everyday terms, how Janet and John’s journey times compare. \[2\]

2 Two independent assessors awarded marks to each of 5 projects. The results were as shown in the table.

<table>
<thead>
<tr>
<th>Project</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>First assessor</td>
<td>38</td>
<td>91</td>
<td>62</td>
<td>83</td>
<td>61</td>
</tr>
<tr>
<td>Second assessor</td>
<td>56</td>
<td>84</td>
<td>41</td>
<td>85</td>
<td>62</td>
</tr>
</tbody>
</table>

(i) Calculate Spearman’s rank correlation coefficient for the data. \[5\]

(ii) Show, by sketching a suitable scatter diagram, how two assessors might have assessed 5 projects in such a way that Spearman’s rank correlation coefficient for their marks was +1 while the product moment correlation coefficient for their marks was not +1. (Your scatter diagram need not be drawn accurately to scale.) \[2\]

3 Five friends, Ali, Bev, Carla, Don and Ed, stand in a line for a photograph.

(i) How many different possible arrangements are there if Ali, Bev and Carla stand next to each other? \[2\]

(ii) How many different possible arrangements are there if none of Ali, Bev and Carla stand next to each other? \[3\]

(iii) If all possible arrangements are equally likely, find the probability that two of Ali, Bev and Carla are next to each other, but the third is not next to either of the other two. \[3\]

4 Each packet of the breakfast cereal Fizz contains one plastic toy animal. There are five different animals in the set, and the cereal manufacturers use equal numbers of each. Without opening a packet it is impossible to tell which animal it contains. A family has already collected four different animals at the start of a year and they now need to collect an elephant to complete their set. The family is interested in how many packets they will need to buy before they complete their set.

(i) Name an appropriate distribution with which to model this situation. State the value(s) of any parameter(s) of the distribution, and state also any assumption(s) needed for the distribution to be a valid model. \[3\]

(ii) Find the probability that the family will complete their set with the third packet they buy after the start of the year. \[2\]

(iii) Find the probability that, in order to complete their collection, the family will need to buy more than 4 packets after the start of the year. \[3\]
A sixth-form class consists of 7 girls and 5 boys. Three students from the class are chosen at random. The number of boys chosen is denoted by the random variable \( X \). Show that

(i) \( P(X = 0) = \frac{7}{44} \). [2]

(ii) \( P(X = 2) = \frac{7}{22} \). [3]

The complete probability distribution of \( X \) is shown in the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(X = x) )</td>
<td>( \frac{7}{44} )</td>
<td>( \frac{21}{44} )</td>
<td>( \frac{7}{22} )</td>
<td>( \frac{1}{22} )</td>
</tr>
</tbody>
</table>

(iii) Calculate \( E(X) \) and \( Var(X) \). [5]

The diagram shows the cumulative frequency graphs for the marks scored by the candidates in an examination. The 2000 candidates each took two papers; the upper curve shows the distribution of marks on paper 1 and the lower curve shows the distribution on paper 2. The maximum mark on each paper was 100.

(i) Use the diagram to estimate the median mark for each of paper 1 and paper 2. [3]

(ii) State with a reason which of the two papers you think was the easier one. [2]

(iii) To achieve grade A on paper 1 candidates had to score 66 marks out of 100. What mark on paper 2 gives equal proportions of candidates achieving grade A on the two papers? What is this proportion? [4]

(iv) The candidates’ marks for the two papers could also be illustrated by means of a pair of box-and-whisker plots. Give two brief comments comparing the usefulness of cumulative frequency graphs and box-and-whisker plots for representing the data. [2]
7 Items from a production line are examined for any defects. The probability that any item will be found to be defective is 0.15, independently of all other items.

(i) A batch of 16 items is inspected. Using tables of cumulative binomial probabilities, or otherwise, find the probability that

(a) at least 4 items in the batch are defective, [2]
(b) exactly 4 items in the batch are defective. [2]

(ii) Five batches, each containing 16 items, are taken.

(a) Find the probability that at most 2 of these 5 batches contain at least 4 defective items. [4]
(b) Find the expected number of batches that contain at least 4 defective items. [2]

8 An experiment was conducted to see whether there was any relationship between the maximum tidal current, \( y \) cm s\(^{-1}\), and the tidal range, \( x \) metres, at a particular marine location. [The tidal range is the difference between the height of high tide and the height of low tide.] Readings were taken over a period of 12 days, and the results are shown in the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>2.0</th>
<th>2.4</th>
<th>3.0</th>
<th>3.1</th>
<th>3.4</th>
<th>3.7</th>
<th>3.8</th>
<th>3.9</th>
<th>4.0</th>
<th>4.5</th>
<th>4.6</th>
<th>4.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>15.2</td>
<td>22.0</td>
<td>25.2</td>
<td>33.0</td>
<td>33.1</td>
<td>34.2</td>
<td>51.0</td>
<td>42.3</td>
<td>45.0</td>
<td>50.7</td>
<td>61.0</td>
<td>59.2</td>
</tr>
</tbody>
</table>

\( \sum x = 43.3, \sum y = 471.9, \sum x^2 = 164.69, \sum y^2 = 20915.75, \sum xy = 1837.78. \]

The scatter diagram below illustrates the data.

(i) Calculate the product moment correlation coefficient for the data, and comment briefly on your answer with reference to the appearance of the scatter diagram. [4]

(ii) Calculate the equation of the regression line of maximum tidal current on tidal range. [4]

(iii) Estimate the maximum tidal current on a day when the tidal range is 4.2 m, and comment briefly on how reliable you consider your estimate is likely to be. [3]

(iv) It is suggested that the equation found in part (ii) could be used to predict the maximum tidal current on a day when the tidal range is 15 m. Comment briefly on the validity of this suggestion. [2]
### Specimen Paper 1

<table>
<thead>
<tr>
<th>Question</th>
<th>Description</th>
<th>Marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (i)</td>
<td>Mean is 84.8 minutes</td>
<td>B1</td>
</tr>
<tr>
<td></td>
<td>Standard deviation = [\sqrt{\frac{1800+44}{25}} = 84.8^2] minutes</td>
<td>M1</td>
</tr>
<tr>
<td></td>
<td>= 3.27 minutes</td>
<td>A1</td>
</tr>
<tr>
<td>(ii)</td>
<td>John’s average time is about 5 minutes less than Janet’s</td>
<td>B1</td>
</tr>
<tr>
<td></td>
<td>John’s times are more variable than Janet’s</td>
<td>B1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Question</th>
<th>Description</th>
<th>Marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 (i)</td>
<td>Ranks are: 1 5 3 4 2</td>
<td>B2</td>
</tr>
<tr>
<td></td>
<td>Values of (d) are (-1, 1, 2, -1, -1)</td>
<td>M1</td>
</tr>
<tr>
<td></td>
<td>(r = 1 - \frac{6 \times 8}{5 \times 24} = 0.6)</td>
<td>M1</td>
</tr>
<tr>
<td>(ii)</td>
<td>For five points, showing any non-linear ‘increasing’ relationship</td>
<td>B2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Question</th>
<th>Description</th>
<th>Marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 (i)</td>
<td>(3! \times 3! = 36)</td>
<td>M1</td>
</tr>
<tr>
<td></td>
<td>For at least one factor of 3!</td>
<td>A1</td>
</tr>
<tr>
<td>(ii)</td>
<td>Ali, Bev and Carla must be in 1st, 3rd, 5th, posns</td>
<td>B1</td>
</tr>
<tr>
<td></td>
<td>Hence number of ways is (3! \times 2! = 12)</td>
<td>M1</td>
</tr>
<tr>
<td></td>
<td>For identifying this restriction</td>
<td>A1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Question</th>
<th>Description</th>
<th>Marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 (i)</td>
<td>Geometric distribution</td>
<td>B1</td>
</tr>
<tr>
<td></td>
<td>(p = \frac{1}{5})</td>
<td>B1</td>
</tr>
<tr>
<td></td>
<td>Each packet is equally likely to contain any of the 5 animals, independently of other packets</td>
<td>B1</td>
</tr>
<tr>
<td>(ii)</td>
<td>(\left(\frac{4}{5}\right)^4 \times \left(\frac{1}{5}\right)^3 = 0.128)</td>
<td>M1</td>
</tr>
<tr>
<td></td>
<td>For any numerical (q^n p^k) calculation</td>
<td>A1</td>
</tr>
<tr>
<td>(iii)</td>
<td>(\left(\frac{4}{5}\right)^4 \text{ or } 1 - \left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^3 \left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^2 \left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^3 \left(\frac{1}{5}\right))</td>
<td>M1</td>
</tr>
<tr>
<td></td>
<td>Allow M mark even if there is an error of 1 in the number of terms</td>
<td>A1</td>
</tr>
<tr>
<td></td>
<td>(\frac{286}{625} \text{ or } 0.4096 \text{ or } 0.410)</td>
<td>A1</td>
</tr>
</tbody>
</table>
5 (i) EITHER: \( P(X = 0) = \binom{7}{3}/\binom{12}{3} = \frac{35}{220} = \frac{7}{44} \)  
\( OR: \) \( P(X = 0) = \frac{7 \times 6 \times 4}{220} = \frac{7}{44} \)

M1 For ratio of relevant \( \binom{n}{r} \) terms  
A1 For showing the given answer correctly

(ii) EITHER: \( P(X = 2) = P(2\text{ boys and } 1\text{ girl}) \)  
\[ = \binom{7}{1} \times \binom{5}{2} \times \binom{12}{3} \]
\[ = \frac{7 \times 10 \times 7}{220} = \frac{7}{22} \]

OR: \( P(X = 2) = P(2\text{ boys and } 1\text{ girl}) \)  
\[ = \frac{5}{12} \times \frac{4}{11} \times \frac{4}{10} \times 3 = \frac{7}{22} \]

M1 For use of three \( \binom{n}{r} \) terms relevant to the 2B, 1G case  
B1 For both \( \binom{5}{2} \) and \( \binom{12}{3} \) correct  
A1 For showing the given answer correctly

(iii) \( E(X) = 0 \times \frac{2}{44} + 1 \times \frac{21}{44} + 2 \times \frac{7}{22} + 3 \times \frac{1}{22} = \frac{5}{4} \)
\( E(X^2) = 0 \times \frac{7}{44} + 1 \times \frac{21}{44} + 4 \times \frac{7}{22} + 9 \times \frac{1}{22} = \frac{95}{44} \)
\( \text{Var}(X) = \frac{95}{44} - \left(\frac{5}{4}\right)^2 = \frac{405}{176} \) or 0.597 (to 3dp)

M1 For correct calculation process  
A1 For correct answer  
B1 For correct numerical expression for \( \sum x^2 p \)  
A1,5 For correct overall method for variance  
A1 For correct answer

6 (i) Medians correspond to 1000 candidates \( m_1 = 38 \), \( m_2 = 63 \)

M1 For reading off at 1000; may be implied  
A1 For correct value for either median  
A1 For both correct

(ii) Paper 2 was easier  
Marks were higher on paper 2

B1 For a correct statement  
B1 For a correct justification

(iii) 66 marks on paper 1 corresponds to 1700 cand, 1700 cand on paper 2 corresponds to 82 marks  
Proportion is \( \frac{2000-1700}{2000} \), i.e. 15%

M1 For reading off at 66; may be implied  
A1 For stating the correct mark  
M1 For relevant subtraction from 2000  
A1 For correct answer 15% or equivalent

(iv) Possible valid comments include:  
Box plots give quick direct comparisons of medians and IQRs  
Box plots don’t include all the information that CF graphs do  
CF graphs can be used to read off values both ways round  
Etc

B1 For any one valid comment  
B1 For any other valid comment

[Turn over]
### 7

|   | (i) |   | (a) | $1 - 0.7899 = 0.210(1)$ | M1 | For complement of relevant tabular value  
|   |     | A1 | 2   | For correct answer  |
|   | (b) |   | 0.9209 - 0.7899 = 0.131 | M1 | For subtracting relevant tabular values  
|   |     | A1 | 2   | For correct answer  |
|   | (ii) | (a) | $0.790^3 + 5 	imes 0.790^2 	imes 0.210 + 10 	imes 0.790^2 	imes 0.210^2$ | M1 | For recognition of $B(5, 0.210)$  
|   |     | M1 | For identification of correct three cases  
|   |     | A1 | √   | For correct expression for the required prob  
|   |     | A1 | 4   | For correct answer  |
|   | (b) |   | Expectation is $5 	imes 0.210 = 1.05$ | M1 | For relevant use of $np$  
|   |     | A1 | 2   | For correct answer  |

### 8

|   | (i) |   | $r = \frac{1837.78 - 43.3 \times 471.9}{\sqrt{(164.69 - 43.3^2)(20915.75 - 471.9^2)}}$ | M1 | For correct formula or calculator use  
|   |     | A1 | For correct value  
|   |     | B1 | For relating the value to 1  
|   |     | B1 | 4   | For a reasonable comment about linearity  |
|   | (ii) |   | Gradient of regression line is | M1 | For correct formula or calculator use  
|   |     | A1 | For correct value for the regression coeff  
|   |     | M1 | For correct form of equn (may be implied)  
|   |     | A1 | 4   | For correct (simplified) equation  |
|   |     |   | $y - \frac{471.9}{12} = 15.9789 \left(x - \frac{43.3}{12}\right)$ | M1 | For substitution into equation from (ii)  
|   |     | A1 | √   | For correct answer  |
|   | (iii) |   | Current is $48.8 \text{ cm s}^{-1}$ | M1 | For identifying extrapolation  
|   |     | A1 | 2   | For correct conclusion  
|   |     | B1 | For any one reasonable comment  |
|   | (iv) |   | As extrapolation is involved, the prediction would be (very) unreliable | M1 | For identifying extrapolation  
|   |     | A1 | 2   | For correct conclusion  |
INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer all the questions.
- Give non-exact numerical answers correct to 3 significant figures, unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphic calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.
- You are reminded of the need for clear presentation in your answers.
1 The standard deviation of a random variable $F$ is 12.0. The mean of $n$ independent observations of $F$ is denoted by $\overline{F}$.

(i) Given that the standard deviation of $\overline{F}$ is 1.50, find the value of $n$. [3]

(ii) For this value of $n$, state, with justification, what can be said about the distribution of $\overline{F}$. [2]

2 A certain neighbourhood contains many small houses (with small gardens) and a few large houses (with large gardens). A sample survey of all houses is to be carried out in this neighbourhood. A student suggests that the sample could be selected by sticking a pin into a map of the neighbourhood the requisite number of times, while blindfolded.

(i) Give two reasons why this method does not produce a random sample. [2]

(ii) Describe a better method. [3]

3 Sixty people each make two throws with a fair six-sided die.

(i) State the probability of one particular person obtaining two sixes. [1]

(ii) Using a suitable approximation, calculate the probability that at least four of the sixty obtain two sixes. [5]

4 The random variable $G$ has mean 20.0 and standard deviation $\sigma$. It is given that $P(G > 15.0) = 0.6$. Assume that $G$ is normally distributed.

(i) (a) Find the value of $\sigma$. [4]

(b) Given that $P(G > g) = 0.4$, find the value of $P(G > 2g)$. [3]

(ii) It is known that no values of $G$ are ever negative. State with a reason what this tells you about the assumption that $G$ is normally distributed. [2]

5 The mean solubility rating of widgets inserted into beer cans is thought to be 84.0, in appropriate units. A random sample of 50 widgets is taken. The solubility ratings, $x$, are summarised by

\[
n = 50, \quad \Sigma x = 4070, \quad \Sigma x^2 = 336100.
\]

Test, at the 5% significance level, whether the mean solubility rating is less than 84.0. [10]
6 On average a motorway police force records one car that has run out of petrol every two days.

(i) Using a Poisson distribution, calculate the probability that, in one randomly chosen day, the police force records exactly two cars that have run out of petrol. [3]

(b) Using a Poisson distribution and a suitable approximation to the binomial distribution, calculate the probability that, in one year of 365 days, there are fewer than 205 days on which the police force records no cars that have run out of petrol. [6]

(ii) State an assumption needed for the Poisson distribution to be appropriate in part (i), and explain why this assumption is unlikely to be valid. [2]

7 The time, in minutes, for which a customer is prepared to wait on a telephone complaints line is modelled by the random variable \( X \). The probability density function of \( X \) is given by

\[
f(x) = \begin{cases} 
  kx(9-x^2) & 0 \leq x \leq 3, \\
  0 & \text{otherwise}, 
\end{cases}
\]

where \( k \) is a constant.

(i) Show that \( k = \frac{4}{31} \). [2]

(ii) Find \( E(X) \). [3]

(iii) (a) Show that the value \( y \) which satisfies \( P(X < y) = \frac{1}{5} \) satisfies

\[
5y^4 - 90y^2 + 243 = 0.
\]

(b) Using the substitution \( w = y^2 \), or otherwise, solve the equation in part (a) to find the value of \( y \). [3]

8 The proportion of left-handed adults in a country is known to be 15%. It is suggested that for mathematicians the proportion is greater than 15%. A random sample of 12 members of a university mathematics department is taken, and it is found to include five who are left-handed.

(i) Stating your hypotheses, test whether the suggestion is justified, using a significance level as close to 5% as possible. [8]

(ii) In fact the significance test cannot be carried out at a significance level of exactly 5%. State the probability of making a Type I error in the test. [2]

(iii) Find the probability of making a Type II error in the test for the case when the proportion of mathematicians who are left-handed is actually 20%. [2]

(iv) Determine, as accurately as the tables of cumulative binomial probabilities allow, the actual proportion of mathematicians who are left-handed for which the probability of making a Type II error in the test is 0.01. [2]
<table>
<thead>
<tr>
<th></th>
<th>(i)</th>
<th>(ii)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{12.0}{\sqrt{n}} = 1.50 \Rightarrow \sqrt{n} = \frac{12.0}{1.50} = 8 \Rightarrow n = 64 )</td>
<td>( n ) is large, the distribution of ( F ) can be taken to be normal, according to the Central Limit Theorem</td>
<td>B1</td>
<td>For any correct equation involving ( n )</td>
<td>M1</td>
</tr>
<tr>
<td>2</td>
<td>Reasons for bias may include: Larger properties more likely to be picked Some regions of the map more/less likely</td>
<td>Make a list of all the houses in the neighbourhood Number the houses from 1 upwards Select the sample using random numbers</td>
<td>B1</td>
<td>For stating one valid relevant reason</td>
<td>B1</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{1}{36} )</td>
<td>Number obtaining two sixes ( \sim B(60, \frac{1}{36}) ) Approximate distribution is ( \text{Po}(\frac{5}{3}) ) ( P(\geq 4) = 1 - e^{-\frac{5}{3}} \left[ 1 + \frac{5}{3} + \frac{(5/3)^2}{2!} + \frac{(5/3)^3}{3!} \right] ) [ = 0.0883 ]</td>
<td>B1</td>
<td>For correct probability</td>
<td>M1</td>
</tr>
<tr>
<td>4</td>
<td>(a) ( \frac{15.0-20.0}{\sigma} = -0.253 )</td>
<td>[ \text{Hence } \sigma = \frac{5}{0.253} = 19.8 ]</td>
<td>M1</td>
<td>For standardising and equating to ( \Phi^{-1}(p) )</td>
<td>B1</td>
</tr>
<tr>
<td></td>
<td>(b) ( g = 25.0 ), using symmetry</td>
<td>[ \text{Hence } P(G &gt; 2g) = 1 - \Phi \left( \frac{50.0-20.0}{19.8} \right) ] [ = 1 - 0.935 = 0.065 ]</td>
<td>B1</td>
<td>For stating (or finding) the value of ( g )</td>
<td>M1</td>
</tr>
<tr>
<td>5</td>
<td>If normal, ( P(G &lt; 0) ) is substantial Hence the assumption seems unjustified</td>
<td></td>
<td>M1</td>
<td>For considering relevant normal probability</td>
<td>A1</td>
</tr>
</tbody>
</table>
5 \[
\bar{x} = \frac{4070}{50} = 81.4 \\
\sigma^2 = \frac{336100}{49} - \frac{4070^2}{49 \times 50} = 98 \\
H_0: \mu = 84.0; \quad H_1: \mu < 84.0
\]

**Either:** 
\[
z = \frac{\bar{x} - 84.0}{\sqrt{\sigma^2/50}} = -1.857
\]

This is significant, since \(-1.857 < -1.645\)

**Or:** 
\[
c = \frac{84.0}{\sqrt{\sigma^2/50}} = -1.645 \Rightarrow c = 81.697
\]

\(\bar{x}\) is in the critical region since 81.4 < 81.697

Hence \(H_0\) is rejected

There is sufficient evidence to conclude that the mean solubility rating is less than 84.0

---

6

(i) (a) For one day, the distribution is \(P(0.5)\)

Hence \(P(\text{exactly 2}) = 0.9856 - 0.9098 = 0.0758\)

(b) No. of days with no cars \(\sim B(365, 0.6065)\)

Normal approximation is \(N(221.3725, 87.11)\)

\(P(< 205) = P\left( Z < \frac{204.5 - 221.3725}{\sqrt{87.11}} \right) = 0.0353\)

(ii) Events (cars running out of petrol) must occur at a constant average rate. This seems unlikely, given that there will be different volumes of traffic on different days of the week (e.g. weekdays and weekends)
7 (i) \[ 1 = k \int_0^3 (9x - x^3) \, dx = k \left[ \frac{3}{2} x^2 - \frac{1}{4} x^4 \right]_0^3 = \frac{k}{4} \]

Hence \( k = \frac{4}{3 \times 8} \)

M1 For equating to 1 and integrating

A1 2 For showing given answer correctly

(ii) \[ E(X) = \frac{4}{3 \times 8} \int_0^3 x^2 (9 - x^3) \, dx = \frac{4}{3 \times 8} \left[ \frac{3}{2} x^3 - \frac{1}{5} x^5 \right]_0^3 = 1.6 \]

M1 For attempt at \( \int f(x) \, dx \)

A1 For correct indefinite integral, in any form

A1 3 For correct answer 1.6

(iii) (a) \[ \frac{3}{5} = \int_0^y x(9 - x^3) \, dx = \int_0^y \left[ \frac{3}{2} x^2 - \frac{1}{4} x^4 \right] \]

Hence \( \frac{3}{5} = \frac{4}{3 \times 8} \left[ \frac{9}{2} y^2 - \frac{1}{4} y^4 \right] \)

i.e. \( 5 y^4 - 90 y^2 + 243 = 0 \)

M1 For attempt at \( \int f(x) \, dx = \frac{3}{5} \)

B1 For correct indefinite integral, in any form

M1 Use limits to produce relevant equation in \( y \)

A1 4 For showing given answer correctly

(b) \[ w = \frac{90 \pm \sqrt{(90^2 - 4 \times 5 \times 243)}}{10} = 3.31 \text{ or } 14.7 \]

M1 For use of quadratic formula to find \( w \)

A1 For either value found correctly

A1 3 For correct (unique) answer 1.82

8 (i) \( H_0 : p = 0.15; \ H_1 : p > 0.15 \)

Under \( H_0 \), number left-handed \( L \sim B(12, 0.15) \)

\[ P(L \geq 5) = 1 - 0.9761 = 0.0239 \]

M1 For correct distribution stated or implied

M1 For calculation of relevant tail probability, or finding the critical region

A1 For correct value 0.0239 or region \( l \geq 5 \)

This is significant, since 0.0239 < 0.05

Hence \( H_0 \) is rejected

A1 For stating or implying rejection of \( H_0 \)

Accept the suggestion that the proportion of mathematicians who are left-handed is more than 15%

A1 8 For stating the outcome in context

(ii) \( P_1 = P(L \text{ in critical region}) = 0.0239 \)

M1 For evaluating \( P(\text{reject } H_0) \)

A1 2 For correct answer 0.0239 or equivalent

(iii) \( P_2 = P(L \leq 4 \mid p = 0.2) = 0.9274 \)

M1 For evaluating \( P(\text{accept } H_0) \) with \( p = 0.2 \)

A1 2 For correct probability

(iv) \( P_3 = 0.0188 \) for \( p = 4 \) and 0.0095 for \( p = 0.7 \)

So the proportion is between 67% and 70%

M1 For relevant use of tables

A1 2 For an appropriate conclusion
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• You are reminded of the need for clear presentation in your answers.
A car repair firm receives call-outs both as a result of breakdowns and also as a result of accidents. On weekdays (Monday to Friday), call-outs resulting from breakdowns occur at random, at an average rate of 6 per 5-day week; call-outs resulting from accidents occur at random, at an average rate of 2 per 5-day week. The two types of call-out occur independently of each other. Find the probability that the total number of call-outs received by the firm on one randomly chosen weekday is more than 3. [5]

Boxes of matches contain 50 matches. Full boxes have mean mass 20.0 grams and standard deviation 0.4 grams. Empty boxes have mean mass 12.5 grams and standard deviation 0.2 grams. Stating any assumptions that you need to make, calculate the mean and standard deviation of the mass of a match. [7]

A random sample of 80 precision-engineered cylindrical components is checked as part of a quality control process. The diameters of the cylinders should be 25.00 cm. Accurate measurements of the diameters, x cm, for the sample are summarised by
\[
\Sigma(x - 25) = 0.44, \quad \Sigma(x - 25)^2 = 0.2287.
\]

(i) Calculate a 99% confidence interval for the population mean diameter of the components. [6]

(ii) For the calculation in part (i) to be valid, is it necessary to assume that component diameters are normally distributed? Justify your answer. [2]

The lengths of time, in seconds, between vehicles passing a fixed observation point on a road were recorded at a time when traffic was flowing freely. The frequency distribution in Table 1 is a summary of the data from 100 observations.

<table>
<thead>
<tr>
<th>Time interval (x seconds)</th>
<th>0 &lt; x ≤ 5</th>
<th>5 &lt; x ≤ 10</th>
<th>10 &lt; x ≤ 20</th>
<th>20 &lt; x ≤ 40</th>
<th>40 &lt; x</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observed frequency</td>
<td>49</td>
<td>22</td>
<td>20</td>
<td>7</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 1

It is thought that the distribution of times might be modelled by the continuous random variable X with probability density function given by
\[
f(x) = \begin{cases} 
0.1 \cdot e^{-0.1x} & \text{if } x > 0, \\
0 & \text{otherwise}.
\end{cases}
\]

Using this model, the expected frequencies (correct to 2 decimal places) for the given time intervals are shown in Table 2.

<table>
<thead>
<tr>
<th>Time interval (x seconds)</th>
<th>0 &lt; x ≤ 5</th>
<th>5 &lt; x ≤ 10</th>
<th>10 &lt; x ≤ 20</th>
<th>20 &lt; x ≤ 40</th>
<th>40 &lt; x</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected frequency</td>
<td>39.35</td>
<td>23.87</td>
<td>23.25</td>
<td>11.70</td>
<td>1.83</td>
</tr>
</tbody>
</table>

Table 2

(i) Show how the expected frequency of 23.87, corresponding to the interval 5 < x ≤ 10, is obtained. [5]

(ii) Test, at the 10% significance level, the goodness of fit of the model to the data. [5]
The continuous random variable \( X \) has a triangular distribution with probability density function given by

\[
f(x) = \begin{cases} 
1 + x & -1 \leq x \leq 0, \\
1 - x & 0 \leq x \leq 1, \\
0 & \text{otherwise.}
\end{cases}
\]

(i) Show that, for \( 0 \leq a \leq 1 \),

\[
P(|X| \leq a) = 2a - a^2. \quad [3]
\]

The random variable \( Y \) is given by \( Y = X^2 \).

(ii) Express \( P(Y \leq y) \) in terms of \( y \), for \( 0 \leq y \leq 1 \), and hence show that the probability density function of \( Y \) is given by

\[
g(y) = \frac{1}{\sqrt{y}} - 1, \quad \text{for } 0 < y \leq 1. \quad [4]
\]

(iii) Use the probability density function of \( Y \) to find \( E(Y) \), and show how the value of \( E(Y) \) may also be obtained directly using the probability density function of \( X \). \quad [4]

(iv) Find \( E(\sqrt{Y}) \). \quad [2]

Certain types of food are now sold in metric units. A random sample of 1000 shoppers was asked whether they were in favour of the change to metric units or not. The results, classified according to age, were as shown in the table.

<table>
<thead>
<tr>
<th>Age of shopper</th>
<th>Under 35</th>
<th>35 and over</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>In favour of change</td>
<td>187</td>
<td>161</td>
<td>348</td>
</tr>
<tr>
<td>Not in favour of change</td>
<td>283</td>
<td>369</td>
<td>652</td>
</tr>
<tr>
<td>Total</td>
<td>470</td>
<td>530</td>
<td>1000</td>
</tr>
</tbody>
</table>

(i) Use a \( \chi^2 \) test to show that there is very strong evidence that shoppers’ views about changing to metric units are not independent of their ages. \quad [7]

(ii) The data may also be regarded as consisting of two random samples of shoppers; one sample consists of 470 shoppers aged under 35, of whom 187 were in favour of change, and the second sample consists of 530 shoppers aged 35 or over, of whom 161 were in favour of change. Determine whether a test for equality of population proportions supports the conclusion in part (i). \quad [7]
A factory manager wished to compare two methods of assembling a new component, to determine which method could be carried out more quickly, on average, by the workforce. A random sample of 12 workers was taken, and each worker tried out each of the methods of assembly. The times taken, in seconds, are shown in the table.

<table>
<thead>
<tr>
<th>Worker</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>J</th>
<th>K</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time in seconds for Method 1</td>
<td>48</td>
<td>38</td>
<td>47</td>
<td>59</td>
<td>62</td>
<td>41</td>
<td>50</td>
<td>52</td>
<td>58</td>
<td>54</td>
<td>49</td>
<td>60</td>
</tr>
<tr>
<td>Time in seconds for Method 2</td>
<td>47</td>
<td>40</td>
<td>38</td>
<td>55</td>
<td>57</td>
<td>42</td>
<td>42</td>
<td>40</td>
<td>62</td>
<td>47</td>
<td>47</td>
<td>51</td>
</tr>
</tbody>
</table>

(i) (a) Carry out an appropriate $t$-test, using a 2% significance level, to test whether there is any difference in the times for the two methods of assembly. [8]

(b) State an assumption needed in carrying out this test. [1]

(c) Calculate a 95% confidence interval for the population mean time difference for the two methods of assembly. [3]

(ii) Instead of using the same 12 workers to try both methods, the factory manager could have used two independent random samples of workers, allocating Method 1 to the members of one sample and Method 2 to the members of the other sample.

(a) State one disadvantage of a procedure based on two independent random samples. [1]

(b) State any assumptions that would need to be made to carry out a $t$-test based on two independent random samples. [2]
1. Model for call-outs is Poisson
   Mean is \( \frac{1}{5}(6 + 2) \)
   = 1.6
   Probability is \( 1 - 0.9212 \)
   = 0.0788

   - For any implication of Poisson
   - For summing two relevant parameters
   - For correct mean of 1.6
   - For relevant use of tables
   - For correct answer

2. Assume \( F = E + M_1 + M_2 + \ldots + M_{50} \), where
   - the masses of the 50 matches in a box are independent
   - the mass of the empty box is independent of the masses of the matches
   \( 20.0 = 12.5 + 50\mu \)
   Hence mean mass of a match is 0.15 grams
   \( 0.4^2 = 0.2^2 + 50\sigma^2 \)
   Hence standard deviation is 0.049 grams

   - For one relevant valid assumption
   - For another relevant valid assumption
   - For attempting \( E(F) \) in terms of \( \mu \)
   - For correct value 0.15
   - For attempting \( \text{Var}(F) \) as a sum
   - For correct equation
   - For correct value 0.049

3. (i) \( \bar{x} = 25.0055 \)
   \[ s^2 = \frac{1}{79} \left( 0.2287 - \frac{0.44^2}{80} \right) = 0.00286 \]
   Interval is \( 25.0055 \pm 2.576 \sqrt{\frac{0.00286}{80}} \)
   Hence \( 24.99(0) < \mu < 25.02(1) \)

   - For correct sample mean, or equivalent; the 25 may be taken into account later
   - For correct unsimplified expression
   - For correct unbiased estimate
   - For calculation of the form \( \bar{x} \pm z \sqrt{s^2/n} \)
   - For relevant use of \( z = 2.576 \)
   - For correct interval, stated to an appropriate degree of accuracy

(ii) The sample size of 80 is sufficient large for the Central Limit Theorem to apply, so it is not necessary to assume a normal distribution

   - For mention of sample size and CLT
   - For the correct conclusion and reason

4. (i) \( f_e = 100 \times \int_{\frac{1}{5}}^{10} 0.1 e^{-0.1x} \, dx \)
   \( = 100 \left[ -e^{-0.1x} \right]_{\frac{1}{5}}^{10} \)
   \( = 100(e^{-0.5} - e^{-1}) = 23.87 \)

   - For attempting to integrate \( f(x) \)
   - For correct indefinite integral
   - For multiplying by total frequency
   - For use of correct limits
   - For obtaining given answer correctly

(ii) Combining: \( f_o \) and \( f_e \)

   Test statistic is \( 9.65^2 + 1.87^2 + 3.25^2 + 4.53^2 = 4.484 \)
   This is less than 6.251
   Hence there is a satisfactory fit

   - For combining the last two classes
   - For correct calculation process
   - For use of correct limits
   - For comparison with the correct critical value
   - For correct conclusion, in terms of the fit
5 (i) \[ P(X < a) = P(-a < X < a) \]
\[ = \int_{-a}^{0} (1 + x) \, dx + \int_{0}^{a} (1 - x) \, dx \]
\[ = \left[ x + \frac{1}{2} x^2 \right]_{-a}^{0} + \left[ x - \frac{1}{2} x^2 \right]_{0}^{a} = 2a - a^2 \]
\[ \text{M1} \quad \text{For consideration of two areas, or equival} \]
\[ \text{A1} \quad \text{For integrals or equivalent trapezia} \]
\[ \text{A1} \quad \text{For showing the given answer correctly} \]

(ii) \[ P(Y \leq y) = P(X^2 \leq y) = P\left(\left| X \right| \leq \sqrt{y} \right) = 2\sqrt{y} - y \]
Hence the pgf of \( Y \) is \[ \frac{d}{dy} \left( 2\sqrt{y} - y \right) = \frac{1}{\sqrt{y}} - 1 \]
\[ \text{M1} \quad \text{For expression of} \ P(X^2 \leq y) \text{in terms of} \ y \]
\[ \text{A1} \quad \text{For correct expression} \ 2\sqrt{y} - y \]
\[ \text{M1} \quad \text{For differentiation of previous expression} \]
\[ \text{A1} \quad \text{For showing the given answer correctly} \]

(iii) \[ E(Y) = \int_{0}^{1} y^2 \, dy = \left[ \frac{y^3}{3} \right]_{0}^{1} = \frac{1}{3} \]
\[ E(X^2) = \int_{0}^{1} (x^2 + x^3) \, dx + \int_{0}^{1} (x^2 - x^3) \, dx \]
\[ = \left[ \frac{1}{3} x^3 + \frac{1}{4} x^4 \right]_{0}^{1} + \left[ \frac{1}{2} x^2 - \frac{1}{4} x^4 \right]_{0}^{1} = \frac{1}{12} + \frac{1}{12} = \frac{1}{6} \]
\[ \text{M1} \quad \text{For the correct integral in terms of} \ y \]
\[ \text{A1} \quad \text{For correct answer} \ \frac{1}{6} \]
\[ \text{A1} \quad \text{For the correct integrals in terms of} \ x \]
\[ \text{A1} \quad \text{For the correct answer correctly obtained} \]

(iv) \[ E(Y) = \int_{0}^{1} y^2 \, g(y) \, dy = \int_{0}^{1} (1 - y^2) \, dy \]
\[ = \left[ y - \frac{1}{3} y^3 \right]_{0}^{1} = \frac{1}{3} \]
\[ \text{M1} \quad \text{For forming the correct integral} \]
\[ \text{A1} \quad \text{For the correct answer} \ \frac{1}{3} \]

6 (i) \[ H_0 : \text{shoppers’ views and age are independent}, \]
\[ H_1 : \text{shoppers’ views and age are not independent} \]
Exp frequencies under \( H_0 \) are
\[ \begin{array}{ccc}
0.163 & 0.356 & 0.1844 \\
0.306 & 0.345 & 0.56 \\
\end{array} \]
\[ \text{M1} \quad \text{For correct method for expected frequencies} \]
\[ \text{A1} \quad \text{For all four correct} \]
\[ \text{Test statistic is} \quad \frac{22.94^2}{163.56} + \frac{22.94^2}{184.44} + \frac{22.94^2}{306.44} + \frac{22.94^2}{345.56} = 9.31... \]
\[ \text{M1} \quad \text{For correct calculation process, inc Yates correction} \]
\[ \text{A1} \quad \text{For correct value of the test statistic} \]
This is greater than the critical 0.5% value of 7.879
Hence there is very strong evidence to reject \( H_0 \)
and conclude that views about changing to metric units are not independent of age
\[ \text{A1} \quad \text{For a relevant (1 df) comparison} \]
\[ \text{A1} \quad \text{For correctly justifying the given answer (the final two marks remain available if Yates’ correction is omitted)} \]

(ii) \[ H_0 : p_1 = p_2, \quad H_1 : p_1 \neq p_2 \]
Under \( H_0 \) the sample value of the common proportion is
\[ \frac{187+161}{1000} = 0.348 \]
\[ \text{B1} \quad \text{For both hypotheses stated} \]
\[ \text{B1} \quad \text{For correct value of estimated} \ p \]
\[ \text{M1} \quad \text{For num} \ p_1 - p_2 \text{and denon using attempted s.d. based on a common estimate of} \ p \]
\[ \text{A1} \quad \text{For completely correct expression} \]
\[ \text{A1} \quad \text{For correct value of the test statistic} \]
\[ \text{M1} \quad \text{For a relevant comparison using the normal distribution} \]
\[ \text{A1} \quad \text{For any relevant comparison or comment} \]
\[ \text{A1} \quad \text{For a relevant (1 df) comparison} \]
\[ \text{A1} \quad \text{For correctly justifying the given answer (the final two marks remain available if Yates’ correction is omitted)} \]
(i) 
(a) \( H_0 : \mu_d = 0, \ H_1 : \mu_d \neq 0 \)

\[
\bar{d} = 4.1667
\]

\[
s^2 = \frac{486}{11} - \frac{50^2}{11 \times 12} = 25.2424
\]

Test statistic is \( \frac{4.1667 - 0}{\sqrt{25.2424/12}} = 2.873 \)

This is greater than the critical value 2.718

Hence there is enough evidence to reject \( H_0 \) and conclude that there is a difference between the times for the two methods.

(b) Population of differences is normal

(b) Both samples are from normal populations

The population variances are equal

(ii) 
(a) Variation in the speed of individual workers is not eliminated, and may be large compared with the difference between the methods that is being tested

(b) Both samples are from normal populations

The population variances are equal

<table>
<thead>
<tr>
<th>Question</th>
<th>Marking Scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) (a)</td>
<td>B1</td>
</tr>
<tr>
<td></td>
<td>B1</td>
</tr>
<tr>
<td></td>
<td>M1</td>
</tr>
<tr>
<td></td>
<td>A1</td>
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<td>M1</td>
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<td>A1</td>
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<td>M1</td>
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<td></td>
<td>A1</td>
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<td></td>
<td>8</td>
</tr>
<tr>
<td>(b)</td>
<td>B1</td>
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<td>B1</td>
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<td>A1</td>
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<td>(ii) (a)</td>
<td>B1</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>(b)</td>
<td>B1</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
</tbody>
</table>

15
INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer all the questions.
- Give non-exact numerical answers correct to 3 significant figures, unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphic calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.
- You are reminded of the need for clear presentation in your answers.
1. A continuous random variable $X$ has moment generating function given by

$$M_X(t) = \frac{9}{(3-t)^2}.$$ 

Find the mean and variance of $X$. [5]

2. The events $A$ and $B$ are independent, and $P(A) = P(B) = p$, where $0 < p < 1$.

(i) Express $P(A \cup B)$ in terms of $p$. [3]

(ii) Given that $P((A \cap B) | (A \cup B)) = \frac{1}{2}$, find the value of $P((A \cap B') \cup (A' \cap B))$. [5]

3. A University’s Department of Computing is interested in whether students who have passed A level Mathematics perform better in Computing examinations than those who have not.

A random sample of 19 students was taken from those students who took a particular first year Computing examination. This sample included 12 students who have passed A level Mathematics and 7 students who have not. The marks gained in the Computing examination were as follows:

Students who have passed A level Mathematics: 27, 34, 39, 41, 45, 47, 55, 59, 66, 75, 78, 86.

Students who have not passed A level Mathematics: 17, 21, 28, 35, 37, 54, 64.

Use a suitable non-parametric test to determine if there is evidence, at the 5% significance level, that students who have passed A level Mathematics gain a higher average mark than students who have not passed A level Mathematics. (A normal approximation may be used.) [10]

4. The continuous random variable $X$ has probability density function given by

$$f(x) = \begin{cases} kx & 0 \leq x \leq a, \\ 0 & \text{otherwise} \end{cases}$$

where $k$ is a constant and the value of the parameter $a$ is unknown.

(i) Show that $k = \frac{2}{a^3}$. [2]

The random variable $U$ is defined by $U = \frac{3}{2} X$.

(ii) Show that $U$ is an unbiased estimator of $a$. [3]

(iii) Find, in terms of $a$, the variance of $U$. [4]

The random variable $\lambda X^n$, where $n$ is a positive integer and $\lambda$ is a constant, is an unbiased estimator of $a^n$.

(iv) Express $\lambda$ in terms of $n$. [2]
5 (i) Explain briefly the circumstances under which a non-parametric test of significance should be used in preference to a parametric test. [1]

The acidity of soil can be measured by its pH value. As a part of a Geography project a student measured the pH values of 14 randomly chosen samples of soil in a certain area, with the following results.

5.67 5.73 6.64 6.76 6.10 5.41 5.80 6.52 5.16 5.10 6.71 5.89 5.68 5.37

(ii) Use a Wilcoxon signed-rank test to test whether the average pH value for soil in this area is 6.24. Use a 10% level of significance. [5]

Some time later, the pH values of soil samples taken at exactly the same locations as before were again measured. It was found that, for 3 of the 14 locations, the new pH value was higher than the previous value, while for the other 11 locations the new value was lower.

(iii) Test, at the 5% significance level, whether there is evidence that the average pH value of soil in this area is lower than previously. [5]

6 The joint probability distribution of the discrete random variables $X$ and $Y$ is shown in the following table.

<table>
<thead>
<tr>
<th>$y$</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/6</td>
<td>5/18</td>
</tr>
<tr>
<td>0</td>
<td>2/9</td>
<td>1/3</td>
</tr>
</tbody>
</table>

(i) Show that $E(X) = -\frac{4}{9}$ and find $\text{Var}(X)$. [4]

(ii) Write down the distributions of $X$ conditional on $Y = 2$ and $X$ conditional on $Y = 3$. Find the means of these conditional distributions, and hence verify that

$$E(X) = E(X \mid Y = 2) \times P(Y = 2) + E(X \mid Y = 3) \times P(Y = 3).$$

It is given that $E(Y) = \frac{47}{18}$ and $\text{Var}(Y) = \frac{77}{324}$. [3]

(iii) Find $\text{Cov}(X, Y)$ and state, with a reason, whether $X$ and $Y$ are independent. [4]

(iv) Find $\text{Var}(X + Y)$. [2]
7  The random variable $X$ has a geometric distribution with parameter $p$.

(i)  Show that the probability generating function $G_X(t)$ of $X$ is given by

$$G_X(t) = \frac{pt}{1-t(1-p)}.$$  [3]

(ii) Hence show that $E(X) = \frac{1}{p}$ and that $\text{Var}(X) = \frac{1-p}{p^2}$.  [5]

A child has 4 fair, six-sided dice, one white, one yellow, one blue and one red.

(iii) The child rolls the white die repeatedly until the die shows a six. The number of rolls up to and including the roll on which the white die first shows a six is denoted by $W$. Write down an expression for $G_W(t)$.  [1]

(iv) The child then repeats this process with the yellow die, then with the blue die and then with the red die. By finding an appropriate probability generating function, find the probability that the total number of rolls of the four dice, up to and including the roll on which the red die first shows a six, is exactly 24.  [4]
OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MATHEMATICS

Probability & Statistics 4

MARK SCHEME

Specimen Paper

MAXIMUM MARK 72
### Question 1

**EITHER:**

\[ M_X(t) = \frac{18}{(3-t)^3} \]

Hence \( E(X) = M_X'(0) = \frac{2}{3} \)

\[ M_X(t) = \frac{54}{(3-t)^4} \]

Hence \( \text{Var}(X) = M_X''(0) - [E(X)]^2 = \frac{4}{9} - \frac{4}{9} = \frac{4}{9} \)

**FOR:**

\[ \text{For correct differentiation of the mgf} \]

\[ \text{For correct value for the mean} \]

\[ \text{For correct second derivative} \]

\[ \text{For correct method for the variance} \]

\[ \text{For correct answer} \]

**OR:**

\[ M_X(t) = 1 + \frac{2}{3} t + \frac{4}{9} t^2 + \ldots \]

Hence \( E(X) = \frac{2}{3} \)

\( \text{Var}(X) = (2!) \times \frac{4}{9} - [E(X)]^2 = \frac{2}{3} - \frac{4}{9} = \frac{2}{9} \)

**FOR:**

\[ \text{For attempting binomial expansion of mgf} \]

\[ \text{For first three terms correct (un Simplified)} \]

\[ \text{For correct value for the mean} \]

\[ \text{For correct method for the variance} \]

\[ \text{For correct answer} \]

### Question 2

(i) \( P(A \cup B) = p + p - p \times p = 2p - p^2 \)

**FOR:**

\[ \text{For use of } P(A) + P(B) - P(A \cap B) \]

\[ \text{For } P(A \cap B) = P(A)P(B) \text{ since independent} \]

\[ \text{For correct expression } 2p - p^2 \]

(ii) \[ \frac{p^2}{2p - p^2} = \frac{1}{2} \Rightarrow 2p = 2 - p \Rightarrow p = \frac{2}{3} \]

Hence \( P((A \cap B') \cup (A' \cap B)) = 2 \times \frac{4}{9} \times \frac{1}{3} = \frac{8}{27} \)

**FOR:**

\[ \text{For equation } \frac{P(A \cap B)}{P(A \cup B)} = \frac{1}{2} \]

\[ \text{For solving relevant equation for } p \]

\[ \text{For correct value} \]

\[ \text{For calculation of } 2p(1 - p) \text{ or equivalent} \]

\[ \text{For correct answer } \frac{4}{9} \]

### Question 3

**H_0:** population medians equal, \( H_1: \) higher median for those who passed Mathematics

**Ranking:**

Pass: 3, 5, 8, 9, 10, 11, 13, 14, 16, 17, 18, 19

Not pass: 1, 2, 4, 6, 7, 12, 15

Sum of ranks of those not passing is 47

\[ R_n \sim N(\frac{1}{2} \times 7 \times 20, \frac{1}{12} \times 7 \times 12 \times 20) = N(70, 140) \]

**EITHER:**

Test statistic is \[ \frac{47.5 - 70}{\sqrt{140}} = -1.902 \]

This is less than \(-1.645\)

**FOR:**

\[ \text{For both hypotheses stated correctly} \]

\[ \text{For attempt at ranking correctly} \]

\[ \text{For correct sum of ranks} \]

\[ \text{For using the appropriate normal approx} \]

\[ \text{For both parameters correct} \]

\[ \text{For standardising} \]

\[ \text{For correct value of test statistic (allow A1 if correct apart from missing or wrong c.c.)} \]

\[ \text{For comparison with correct critical value} \]

**OR:**

Critical region is \[ \frac{X + 0.5 - 70}{\sqrt{140}} < -1.645 \]

i.e. \( X \leq 50 \)

Sample value 47 lies in the critical region

Hence there is evidence that those passing Mathematics have a higher average score

**FOR:**

\[ \text{For conclusion stated in context} \]

\[ \text{For both hypotheses stated correctly} \]

\[ \text{For attempt at ranking correctly} \]

\[ \text{For correct sum of ranks} \]

\[ \text{For using the appropriate normal approx} \]

\[ \text{For both parameters correct} \]

\[ \text{For standardising} \]

\[ \text{For correct value of test statistic (allow A1 if correct apart from missing or wrong c.c.)} \]

\[ \text{For comparison with correct critical value} \]

\[ \text{For setting up the appropriate inequality} \]

\[ \text{For correct critical region (allow A1 if correct apart from missing or wrong c.c.)} \]

\[ \text{For comparing 47 with critical region} \]

\[ \text{For conclusion stated in context} \]
4 (i) \( \int_0^a kx \, dx = 1 \Rightarrow \frac{1}{2}ka^2 = 1 \Rightarrow k = \frac{2}{a^2} \)

M1 For use of \( \int_0^a f(x) \, dx = 1 \)

A1 For showing the given answer correctly

(ii) \( E(U) = \frac{2}{a} \int_0^a kx^2 \, dx = \frac{2}{a} \times \frac{1}{4}ka^3 = a \)

B1 For stating or implying \( E(U) = \frac{2}{a} E(X) \)

M1 For use of \( \int_0^a x f(x) \, dx \)

Hence \( U \) is an unbiased estimator of \( a \)

A1 For showing the given result correctly

(iii) \( E(U^2) = \int_0^a \left( \frac{1}{2}x \right)^2 kx \, dx = \frac{9}{16} ka^4 = \frac{9}{8} a^2 \)

M1 For correct process for \( E(U^2) \)

A1 For correct value \( \frac{9}{8} a^2 \)

Hence \( \text{Var}(U) = \frac{9}{8} a^2 - a^2 = \frac{1}{8} a^2 \)

M1 For correct process for \( \text{Var}(U) \)

A1\(^\wedge\) 4 For correct answer

(Alternatively via \( \text{Var}(U) = \frac{9}{8} \text{Var}(X) \).)

(iv) \( \frac{2A}{a^2} \int_0^a x^{n+1} \, dx = a^n \Rightarrow \frac{2A}{a^2} \times \frac{a^{n+2}}{n+2} = a^n \)

M1 For using \( \lambda E(X^n) = a^n \)

A1 For correct answer

5 (i) A non-parametric test is needed when there is no information (or reasonable assumption) available about an underlying distribution

B1 1 For a correct statement

(ii) \( H_0 : \) population median pH is 6.24, \( H_1 : \) population median pH is not 6.24

Deviations from NH value 6.24 are:

\(-0.57 -0.51 0.40 0.52 -0.14 -0.83 -0.44 = -0.28 -1.08 -1.14 0.47 -0.35 -0.56 -0.87\)

Signed ranks are:

\(-10 -7 4 8 -1 -11 -5 2 -13 -14 6 -3 -9 -12\)

M1 For calculating signed differences from 6.24

M1 For calculating signed ranks

Test statistic is \( 2 + 4 + 6 + 8 = 20 \)

A1 For the correct value of the test statistic

M1 For comparing with the correct critical value

A1 6 For correct conclusion based on correct work

(iii) \( H_0 : \) same average pH as before; \( H_1 : \) lower value

\( P(\leq 3 \text{ out of } 14 | H_0) = 0.0287 \)

This is less than 0.05, so we reject \( H_0 \) and conclude that the average pH is now lower

B1 For both hypotheses stated correctly

M1 For relevant use of \( B(14, \frac{1}{2}) \)

A1 For correct value 0.0287

M1 For comparing with 0.05

A1 5 For correct conclusion based on correct work
6 (i) Marginal probabilities for $X$ are $\frac{4}{9}, \frac{5}{9}$.

Hence $E(X) = -1 \times \frac{4}{9} + 0 \times \frac{5}{9} = -\frac{4}{9}$

$\text{Var}(X) = (-1)^2 \times \frac{4}{9} - \left(-\frac{4}{9}\right)^2 = \frac{20}{81}$

(ii) $\begin{array}{c|c|c}
 x & -1 & 0 \\
P_2(X = x) & \frac{3}{4} & \frac{1}{4} \\
\end{array}$

Hence $E(X \mid Y = 2) = -\frac{1}{2}$, $E(X \mid Y = 3) = -\frac{5}{11}$

RHS $= -\frac{4}{11} \times \frac{7}{18} - \frac{5}{11} \times \frac{14}{18} = -\frac{4}{9} = E(X)$

(iii) $E(XY) = -2 \times \frac{1}{6} - 3 \times \frac{5}{18} = -\frac{2}{9}$

$\text{Cov}(X, Y) = -\frac{2}{9} - \left(-\frac{4}{9}\right) \times \frac{42}{18} = -\frac{1}{162}$

$X$ and $Y$ are not independent, as $\text{Cov}(X, Y) \neq 0$

(iv) $\text{Var}(X + Y) = \frac{29}{81} + \frac{37}{324} - \frac{2}{9} = \frac{17}{36}$

7 (i) $G_X(t) = \sum_{r=1}^{\infty} q^{r-1} pt^r$, where $q = 1 - p$

$= pt \sum_{r=1}^{\infty} (qt)^{r-1} = \frac{pt}{1 - qt} = \frac{pt}{1 - (1 - p)t}$

(ii) $G'_X(t) = \frac{p}{(1 - qt)^2}$

Hence $E(X) = G'_X(1) = \frac{p}{p^2} = \frac{1}{p}$

$G''_X(t) = \frac{2pq}{(1 - qt)^3}$

Hence $\text{Var}(X) = G''_X(1) + \frac{1}{p} - \frac{1}{p^2}$

$= \frac{2pq}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{q}{p^2} = \frac{1 - p}{p^2}$

(iii) $G_W(t) = \frac{\frac{1}{6} t}{1 - \frac{5}{6} t}$

(iv) Required pgf is $\left( \frac{\frac{1}{6} t}{1 - \frac{5}{6} t} \right)^4$

Required probability is the coefficient of $t^{24}$

This is $\left( \frac{1}{6} \right)^4 \times (-4) \times (-5) \times (-6) \times \ldots \times (-23) \times \left( \frac{2}{6} \right)^{20}$

$= \frac{20!}{4! \times 20!} = 0.0356$
OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MATHEMATICS 4736

Decision Mathematics 1

Specimen Paper

Additional materials:
Answer booklet
Graph paper
List of Formulae (MF 1)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

• Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
• Answer all the questions.
• Give non-exact numerical answers correct to 3 significant figures, unless a different degree of accuracy is specified in the question or is clearly appropriate.
• You are permitted to use a graphic calculator in this paper.

INFORMATION FOR CANDIDATES

• The number of marks is given in brackets [ ] at the end of each question or part question.
• The total number of marks for this paper is 72.
• Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.
• You are reminded of the need for clear presentation in your answers.
1 The graph $K_5$ has five nodes, $A$, $B$, $C$, $D$ and $E$, and there is an arc joining every node to every other node.

(i) Draw the graph $K_5$ and state how you know that it is Eulerian. [2]

(ii) By listing the arcs involved, give an example of a path in $K_5$. (Your path must include more than one arc.) [1]

(iii) By listing the arcs involved, give an example of a cycle in $K_5$. [1]

2 This question is about a simply connected network with at least three arcs joining 4 nodes. The weights on the arcs are all different and any direct paths always have a smaller weight than the total weight of any indirect paths between two vertices.

(i) Kruskal’s algorithm is used to construct a minimum connector. Explain why the arcs with the smallest and second smallest weights will always be included in this minimum connector. [3]

(ii) Draw a diagram to show that the arc with the third smallest weight need not always be included in a minimum connector. [4]

3 (i) Use the shuttle sort algorithm to sort the list

$$6\quad 3\quad 8\quad 3\quad 2$$

into increasing order. Write down the list that results from each pass through the algorithm. [5]

(ii) Shuttle sort is a quadratic order algorithm. Explain briefly what this statement means. [3]

4 [Answer this question on the insert provided.]

An algorithm involves the following steps.

Step 1: Input two positive integers, $A$ and $B$. Let $C = 0$

Step 2: If $B$ is odd, replace $C$ by $C + A$.

Step 3: If $B = 1$, go to step 6.

Step 4: Replace $A$ by $2A$.

If $B$ is even, replace $B$ by $B + 2$, otherwise replace $B$ by $(B - 1) + 2$.

Step 5: Go back to step 2.

Step 6: Output the value of $C$.

(i) Demonstrate the use of the algorithm for the inputs $A = 6$ and $B = 13$. [5]

(ii) When $B = 8$, what is the output in terms of $A$? What is the relationship between the output and the original inputs? [4]
In this network the vertices represent towns, the arcs represent roads and the weights on the arcs show the shortest distances in kilometres.

(i) The diagram on the insert shows the result of deleting vertex $F$ and all the arcs joined to $F$. Show that a lower bound for the length of the travelling salesperson problem on the original network is $38 \text{ km}$.

The corresponding lower bounds by deleting each of the other vertices are:

- $A: 40 \text{ km}$,
- $B: 39 \text{ km}$,
- $C: 35 \text{ km}$,
- $D: 37 \text{ km}$,
- $E: 35 \text{ km}$.

The route $A\to B\to C\to D\to E\to F\to A$ has length $47 \text{ km}$.

(ii) **Using only this information**, what are the best upper and lower bounds for the length of the solution to the travelling salesperson problem on the network?  

(iii) By considering the orders in which vertices $C$, $D$ and $E$ can be visited, find the best upper bound given by a route of the form $A\to B\to \ldots \to F\to A$.  

[Turn over]
The diagram shows a simplified version of an orienteering course. The vertices represent checkpoints and the weights on the arcs show the travel times between checkpoints, in minutes.

(i) Use Dijkstra's algorithm, starting from checkpoint A, to find the least travel time from A to D. You must show your working, including temporary labels, permanent labels and the order in which permanent labels were assigned. Give the route that takes the least time from A to D. [6]

(ii) By using an appropriate algorithm, find the least time needed to travel every arc in the diagram starting and ending at A. You should show your method clearly. [6]

(iii) Starting from A, apply the nearest neighbour algorithm to the diagram to find a cycle that visits every checkpoint. Use your solution to find a path that visits every checkpoint, starting from A and finishing at D. [3]

Consider the linear programming problem:

maximise $P = 4y - x,$

subject to $x + 4y \leqslant 22,$

$x + y \leqslant 10,$

$-x + 2y \leqslant 8,$

and $x \geqslant 0, \ y \geqslant 0.$

(i) Represent the constraints graphically, shading out the regions where the inequalities are not satisfied. Calculate the value of $x$ and the value of $y$ at each of the vertices of the feasible region. Hence find the maximum value of $P$, clearly indicating where it occurs. [8]

(ii) By introducing slack variables, represent the problem as an initial Simplex tableau and use the Simplex algorithm to solve the problem. [10]

(iii) Indicate on your diagram for part (i) the points that correspond to each stage of the Simplex algorithm carried out in part (ii). [2]
OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MATHEMATICS 4736

Decision Mathematics 1
INSERT for Questions 4, 5 and 6
Specimen Paper

INSTRUCTIONS TO CANDIDATES

- This insert should be used to answer Questions 4, 5 and 6 (i).
- Write your Name, Centre Number and Candidate Number in the spaces provided at the top of this page.
- Write your answers to Questions 4, 5 and 6 (i) in the spaces provided in this insert, and attach it to your answer booklet.

This insert consists of 4 printed pages.
#### (i)

<table>
<thead>
<tr>
<th>STEP</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### (ii)

<table>
<thead>
<tr>
<th>STEP</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

---

4736 Specimen Paper
5 (i) 

(ii) Upper bound = ......................... km
Lower bound = ......................... km

(iii) Best upper bound = ......................... km
Key:
Order of becoming permanent
Permanent value
Temporary values
(do not cross out working)

Least travel time = ........................................ minutes

Route: $A \rightarrow \ldots \rightarrow D$
OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MATHEMATICS
Decision Mathematics 1
MARK SCHEME
Specimen Paper

MAXIMUM MARK 72
1 (i) K₅ is Eulerian since every node is even

(ii) A path is (e.g.) A→B→C

(iii) A cycle is (e.g.) A→B→C→A

2 (i) Using Kruskal’s algorithm, the arc of least weight is chosen first and so is certainly included. The arc of second least weight is chosen next since just two arcs cannot form a cycle

(ii) For any connected graph with 4 nodes and at least 3 arcs

3 (i) 1st pass: 6 3 8 3 2 giving 3 6 8 3 2

   2nd pass: 3 6 8 3 2 giving 3 6 8 3 2

   3rd pass: 3 6 8 3 2

   4th pass: 3 6 8 2

   The number of operations to be carried out, and thus the time to complete the algorithm, is (approximately) proportional to the square of the number of items to be sorted

   (ii) M1 A1 A1 For idea of dependency on ‘size’ of problem

   A1 A1 A1 For number of operations, or time required

   A1 A1 A1 For square of list size
4 (i) | STEP | A | B | C |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>13</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>13</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>24</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>24</td>
<td>3</td>
<td>30</td>
</tr>
<tr>
<td>4</td>
<td>48</td>
<td>1</td>
<td>30</td>
</tr>
<tr>
<td>2</td>
<td>48</td>
<td>1</td>
<td>78</td>
</tr>
<tr>
<td>3</td>
<td>48</td>
<td>1</td>
<td>78</td>
</tr>
<tr>
<td>6</td>
<td>Output 78</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   B1 For assigning value to C in first Step 2
   M1 For updating A and B in first Step 4
   M1 For continuing algorithm and updating C
   A1 For correct new value 30 for C

   A1 5 For correct output

   (ii) | STEP | A | B | C |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>2A</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>4A</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>8A</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>8A</td>
<td>1</td>
<td>8A</td>
</tr>
<tr>
<td>3</td>
<td>8A</td>
<td>1</td>
<td>8A</td>
</tr>
<tr>
<td>6</td>
<td>Output 8A</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The output is the product of the inputs

   B1 4 For identifying multiplication

5 (i) A minimum connector on reduced network has arcs CE, ED, BD, AB, giving length 23 km
Two shortest arcs from F have weights 7, 8
Hence lower bound is 23 + 7 + 8 = 38 km

   M1 For attempt at a relevant minimum connector
   A1 For correct weight 23
   M1 For identifying the two shortest arcs at F
   A1 For showing given answer correctly

   (ii) The best upper bound is 47 km

   The best lower bound is 40 km

   B1 For the correct answer
   B1 2 For the correct answer

   (iii) Other orders are CED, DCE, DEC, ECD, EDC
Shortest is ABDCEFA, of length 42 km

   M1 For calculation of at least one other length
   A1 For any correct bound less than 47 km
   A1 3 For the correct value 42

6 (i) Least travel time is 40 minutes
Route is A–B–C–D

   M1 For correct use of temporary labels
   M1 For updating E and D
   A1 For all permanent labels correct
   B1 For correct order of assignment stated

   B1 6 For correct route

   (ii) The Route Inspection algorithm is used
A, B, C and E are odd nodes
AB = 16  AC = 27  AE = 37
CE = 10  BE = 21  BC = 11

   26 48 48

   Double up on AB and CE
Sum of arcs is 172
Hence shortest time is 172 + 26 = 198 minutes

   M1 For pairing odd nodes correctly
   M1 For selecting appropriate pair for doubling
   M1 For adding weights on all the arcs
   A1 6 For correct value 198

   (iii) Nearest neighbour algorithm gives A–B–C–E–D–A
Hence required path is A–B–C–E–D

   M1 For starting the algorithm correctly, up to C
   A1 For the correct cycle A–B–C–E–D–A
   B1 3 For a correct path
(i) \[ xy + = \text{ for lines } x + 4y = 22 \text{ and } x + y = 10 \]
\[ xy + = \text{ for line } -x + 2y = 8 \]
A1 For correct diagram including shading

B1 For vertices (0, 0), (0, 4), (10, 0)
B1 For vertex (2, 5)
B1 For vertex (6, 4)

Hence maximum \( P = 18 \), occurring at (2, 5)

(ii) For correct pay-off row
M1 For the use of three slack variables
A1 For all constraints correct

Pivot on 2 in row 3
M1 For choice of pivot
M1 For pivoting correctly
A1 For correct tableau

Now pivot on 3 in row 1
M1 For choice of pivot
M1 For pivoting correctly
A1 For correct tableau

Hence \( P = 18 \) when \( x = 2, y = 5 \)
B1 For reading off correctly from final tableau

(iii) Vertices (0, 0) \( \rightarrow \) (0, 4) \( \rightarrow \) (2, 5) indicated
M1 For indication of starting at the origin
A1 For the correct correspondence indicated
INSTRUCTIONS TO CANDIDATES

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- You are reminded of the need for clear presentation in your answers.
1. [Answer this question on the insert provided.]

Six neighbours have decided to paint their houses in bright colours. They will each use a different colour.

- Arthur wants to use lavender, orange or tangerine.
- Bridget wants to use lavender, mauve or pink.
- Carlos wants to use pink or scarlet.
- Davinder wants to use mauve or pink.
- Eric wants to use lavender or orange.
- Ffion wants to use mauve.

Arthur chooses lavender, Bridget chooses mauve, Carlos chooses pink and Eric chooses orange. This leaves Davinder and Ffion with colours that they do not want.

(i) Draw a bipartite graph on the insert, showing which neighbours (A, B, C, D, E, F) want which colours (L, M, O, P, S, T). On a separate diagram on the insert, show the incomplete matching described above. [3]

(ii) By constructing alternating paths obtain the complete matching between the neighbours and the colours. Give your paths and show your matching on the insert. [4]

(iii) Fill in the table on the insert to show how the Hungarian algorithm could have been used to find the complete matching. (You do not need to carry out the Hungarian algorithm.) [2]

2. A company has organised four regional training sessions to take place at the same time in four different cities. The company has to choose four of its five trainers, one to lead each session. The cost (£1000's) of using each trainer in each city is given in the table.

<table>
<thead>
<tr>
<th>City</th>
<th>London</th>
<th>Glasgow</th>
<th>Manchester</th>
<th>Swansea</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adam</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>Betty</td>
<td>3</td>
<td>5</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Clive</td>
<td>3</td>
<td>6</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Dave</td>
<td>2</td>
<td>6</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>Eleanor</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

(i) Convert this into a square matrix and then apply the Hungarian algorithm, reducing rows first, to allocate the trainers to the cities at minimum cost. [7]

(ii) Betty discovers that she is not available on the date set for the training. Find the new minimum cost allocation of trainers to cities. [2]
A flying doctor travels between islands using small planes. Each flight has a weight limit that restricts how much he can carry. A plague has broken out on Farr Island and the doctor needs to take several crates of medical supplies to the island. The crates must be carried on the same planes as the doctor.

The diagram shows a network with (stage; state) variables at the vertices representing the islands, arcs representing flight routes that can be used, and weights on the arcs representing the number of crates that the doctor can carry on each flight.

(i) It is required to find the route from (0; 0) to (3; 0) for which the minimum number of crates that can be carried on any stage is a maximum (the maximin route). The insert gives a dynamic programming tabulation showing stages, states and actions, together with columns for working out the route minimum at each stage and for indicating the current maximin.

Complete the table on the insert sheet and hence find the maximin route and the maximum number of crates that can be carried. [7]

(ii) It is later found that the number of crates that can be carried on the route from (2; 0) to (3; 0) has been recorded incorrectly and should be 15 instead of 5. What is the maximin route now, and how many crates can be carried? [3]
Henry is planning a surprise party for Lucinda. He has left the arrangements until the last moment, so he will hold the party at their home. The table below lists the activities involved, the expected durations, the immediate predecessors and the number of people needed for each activity. Henry has some friends who will help him, so more than one activity can be done at a time.

<table>
<thead>
<tr>
<th>Activity</th>
<th>Duration (hours)</th>
<th>Preceded by</th>
<th>Number of people</th>
</tr>
</thead>
<tbody>
<tr>
<td>A: Telephone other friends</td>
<td>2</td>
<td>–</td>
<td>3</td>
</tr>
<tr>
<td>B: Buy food</td>
<td>1</td>
<td>A</td>
<td>2</td>
</tr>
<tr>
<td>C: Prepare food</td>
<td>4</td>
<td>B</td>
<td>5</td>
</tr>
<tr>
<td>D: Make decorations</td>
<td>3</td>
<td>A</td>
<td>3</td>
</tr>
<tr>
<td>E: Put up decorations</td>
<td>1</td>
<td>D</td>
<td>3</td>
</tr>
<tr>
<td>F: Guests arrive</td>
<td>1</td>
<td>C, E</td>
<td>1</td>
</tr>
</tbody>
</table>

(i) Draw an activity network to represent these activities and the precedences. Carry out forward and reverse passes to determine the minimum completion time and the critical activities. If Lucinda is expected home at 7.00 p.m., what is the latest time that Henry or his friends can begin telephoning the other friends? [7]

(ii) Draw a resource histogram showing time on the horizontal axis and number of people needed on the vertical axis, assuming that each activity starts at its earliest possible start time. What is the maximum number of people needed at any one time? [3]

(iii) Now suppose that Henry’s friends can start buying the food and making the decorations as soon as the telephoning begins. Construct a timetable, with a column for ‘time’ and a column for each person, showing who should do which activity when, in order than the party can be organised in the minimum time using a total of only six people (Henry and five friends). When should the telephoning begin with this schedule? [3]
Fig. 1 shows a directed flow network. The weight on each arc shows the capacity in litres per second.

(i) Find the capacity of the cut $C$ shown. [2]

(ii) Deduce that there is no possible flow from $S$ to $T$ in which both arcs leading into $T$ are saturated. Explain your reasoning clearly. [2]

Fig. 2 shows a possible flow of 160 litres per second through the network.

(iii) On the diagram in the insert, show the excess capacities and potential backflows for this flow. [3]

(iv) Use the labelling procedure to augment the flow as much as possible. Show your working clearly, but do not obscure your answer to part (iii). [4]

(v) Show the final flow that results from part (iv). Explain clearly how you know that this flow is maximal. [3]
6 Rose is playing a game against a computer. Rose aims a laser beam along a row, A, B or C, and, at the same time, the computer aims a laser beam down a column, X, Y or Z. The number of points won by Rose is determined by where the two laser beams cross. These values are given in the table. The computer loses whatever Rose wins.

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

(i) Find Rose’s play-safe strategy and show that the computer’s play-safe strategy is Y. How do you know that the game does not have a stable solution? [3]

(ii) Explain why Rose should never choose row C and hence reduce the game to a 2×3 pay-off matrix. [2]

(iii) Rose intends to play the game a large number of times. She decides to use a standard six-sided die to choose between row A and row B, so that row A is chosen with probability \(a\) and row B is chosen with probability \(1-a\). Show that the expected pay-off for Rose when the computer chooses column X is \(4 - 3a\), and find the corresponding expressions for when the computer chooses columns Y and Z. Sketch a graph showing the expected pay-offs against \(a\), and hence decide on Rose’s optimal choice for \(a\). Describe how Rose could use the die to decide whether to play A or B. [6]

The computer is to choose X, Y and Z with probabilities \(x\), \(y\) and \(z\) respectively, where \(x + y + z = 1\).

Graham is an AS student studying the D1 module. He wants to find the optimal choices for \(x\), \(y\) and \(z\) and starts off by producing a pay-off matrix for the computer.

(iv) Graham produces the following pay-off matrix.

\[
\begin{pmatrix}
3 & 1 & 0 \\
0 & 1 & 2
\end{pmatrix}
\]

Write down the pay-off matrix for the computer and explain what Graham did to its entries to get the values in his pay-off matrix. [2]

(v) Graham then sets up the linear programming problem:

- maximise \(P = p - 4\),
- subject to \(p - 3x - y \leq 0\), \(p - y - 2z \leq 0\), \(x + y + z \leq 1\),
- and \(p \geq 0\), \(x \geq 0\), \(y \geq 0\), \(z \geq 0\).

The Simplex algorithm is applied to the problem and gives \(x = 0.4\) and \(y = 0\). Find the values of \(z\), \(p\) and \(P\) and interpret the solution in the context of the game. [4]
INSTRUCTIONS TO CANDIDATES

- This insert should be used to answer Questions 1, 3 and 5.
- Write your Name, Centre Number and Candidate Number in the spaces provided at the top of this page.
- Write your answers to Questions 1, 3 and 5 in the spaces provided in this insert, and attach it to your answer booklet.
(i) Bipartite graph

(ii) Matching described in question

(iii)
### (i)

<table>
<thead>
<tr>
<th>Stage</th>
<th>State</th>
<th>Action</th>
<th>Route minimum</th>
<th>Current maximin</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
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<tr>
<td>2</td>
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<td>1</td>
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<td>2</td>
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<td>1</td>
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<tr>
<td>0</td>
<td>2</td>
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<td></td>
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</tr>
</tbody>
</table>

Route: .......................................................... ..........................................................

Maximum number of crates that can be carried: ..........................................................

### (ii)

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(i) Capacity of $C$: ........................................................................................................................................

(ii) ................................................................................................................................................................

(iii) ................................................................................................................................................................

(iv) ................................................................................................................................................................

(v) Final flow:

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................................................................................................................................................................
1. (i) For attempt at the bipartite graph
   A1  For correct graph
   B1  For the correct incomplete matching

(ii) Alternating paths are:

   S–C=P–D and T–A=L–B=M–F
   or S–C=P–B=M–D and T–A=L–B=P–D=M–F
   or S–C=P–B=M–F and T–A=L–B=P–D=M–F
   or T–A=L–B=M–D and S–C=P–D=M–F

   M1  For attempt at an alternating path
   A1  For one correct path
   A1  For the second path correct
   B1  For correct matching

(iii) For appropriate zeros and ones (e.g.) to correspond with minimum cost matching
   A1  For a correct table

2. (i) Adding a dummy column gives

   |   |   |   |   |   |   |
   |   |   |   |   |   |
   |   |   |   |   |   |
   |   |   |   |   |   |
   |   |   |   |   |   |
   |   |   |   |   |   |
   |   |   |   |   |   |
   |   |   |   |   |   |
   |   |   |   |   |   |
   |   |   |   |   |   |

   M1  For reducing rows
   M1  For reducing columns
   Four lines are needed to cover zeros
   M1  For covering zeros in the reduced matrix
   A1  For correct augmentation process
   A1  For these two allocations correct
   A1  For any one of the correct possibilities

(ii) Without Betty, reduced matrix is

   |   |   |   |   |   |
   |   |   |   |   |   |
   |   |   |   |   |   |
   |   |   |   |   |   |
   |   |   |   |   |   |
   |   |   |   |   |   |
   |   |   |   |   |   |
   |   |   |   |   |   |
   |   |   |   |   |   |
   |   |   |   |   |   |

   M1  For new reduced matrix
   A1  For either of the correct new possibilities
(i) For dealing with route min column
M1 For at least 6 minima correct
A1 For dealing with maximin column
M1 For Stage 1 section of table all correct
A1 For completely correct table

Route is (0; 0)–(1; 2)–(2; 2)–(3; 0) B1 For correct route

Maximum number of crates is 8 B1 For correct number

(ii) New maximin values are 15, 7, 9, 12, 9, 9, 9 M1 For appropriate re-calculation
Hence new route is (0; 0)–(1; 0)–(2; 0)–(3; 0) A1 For correct new route
New maximum number of crates is 9 A1 For correct number

(iii) B1 For correct arcs and activities (activity on arc network or equivalent with activity at node)
M1 For correct process for forward pass
M1 For correct process for reverse pass
A1 For all early and late times correct
Minimum completion time is 8 hours A1 For correct minimum time stated
Critical activities are A, B, C, F B1 For correct critical activities
Start telephoning at 11.00 am B1 For stating the appropriate time of day

(iii) B1 For resource histogram with axes labelled
A1 For correct heights 3, 3, 8, 8, 8, 5, 1

Maximum number of people needed is 8 B1 For correct number stated

Start telephoning at 10.00 am B1 For correct time stated
(i) Capacity is $150 + 0 + 50 + 80 = 280$ litres/sec

(ii) Maximum flow is $\leq 280$

So flow of $200 + 100 = 300$ is not possible

(iii)

Augment by 70 along $SBCDT$ (e.g.)

New excesses and backflows are as shown above

Now augment by 50 along $SACDT$ (e.g.)

Final excesses and backflows are as shown above

(iv) Augment by 70 along $SBCDT$ (e.g.)

New excesses and backflows are as shown above

Now augment by 50 along $SACDT$ (e.g.)

Final excesses and backflows are as shown above

(v)

The value of the augmented flow is 280 litres/sec, and so is the maximum possible
(i) Play-safe for Rose is B
Play-safe for computer is Y
Not stable as \(-3 + 2 \neq 0\)

(ii) Row C is dominated by row B

(iii) Expected pay-off with X is \(1 \times a + 4(1 - a) = 4 - 3a\)
and with Y is \(3a + 3(1 - a) = 3\)
and with Z is \(4a + 2(1 - a) = 2 + 2a\)

Add 4 to each value

(v) \(z = 0.6\)
\(p = 1.2 \Rightarrow P = -2.8\)
The computer should choose X with probability 0.4 and Z with probability 0.6
On average the computer will lose no more than 2.8 points per game