On the decomposition of risk in life insurance

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1 Motivation

Situation: Life office/policy obtains gain $G_{s,t}$ during $[s,t]$

Bühlmann (1995):

- $G_{s,t}$ must be decomposed into financial and biometric (technical) part
  \[ G_{s,t} = G_{s,t}^F + G_{s,t}^B \]  
  (1)

- $G_{s,t}^B$ must be pooled (diversification, Law of Large Numbers)

- AFIR-problem (Bühlmann, 1995):
  \[(G_{s,t}^F)^+ \text{ belongs to the insured } \Rightarrow \text{ pricing & hedging?} \]

⇒ How to decompose?

⇒ What is “pooling”?

⇒ What is the role of hedging?
2 Model

- Finite discrete time axis $\mathbb{T} = \{0, 1, 2, \ldots, T\}$
- filtered product space $(F, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{F}) \otimes (B, (\mathcal{B}_t)_{t \in \mathbb{T}}, \mathbb{B})$
- $\mathcal{F}_T$, $\mathcal{B}_T$ finite
- $d$ assets with price processes $S = ((S^0_t, \ldots, S^{d-1}_t))_{t \in \mathbb{T}}$
- complete arbitrage-free financial market, unique EMM $\mathbb{Q}$
- $t$-portfolio $\theta = (\theta^0, \ldots, \theta^{d-1})$ is vector of integrable $\mathcal{F}_t \otimes \mathcal{B}_t$-measurable random variables with $t$-value

$$\langle \theta, S_t \rangle = \sum_{j=0}^{d-1} \theta^j S^j_t$$

e.g. zero-coupon bond with maturity 1: $\theta = (1/S^0_1, 0, \ldots, 0)$
3 Market-based valuation

Product measure principle: *market value* (minimum fair price)

\( \pi_t(\theta) \) of \( \theta \) at \( t \in \mathbb{T} \) given by

\[
\pi_t(\theta) = S_t^0 \cdot \mathbb{E}_{Q \otimes B}[\langle \theta, S_T \rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_t].
\] (2)

- \( Q \otimes B \) is one of perhaps many EMM (incomplete market)
- for \( t=0 \): \( \pi_0(\theta) = \mathbb{E}_{Q \otimes B}[\langle \theta, S_T \rangle / S_T^0] = \mathbb{E}_Q[\langle \mathbb{E}_B[\theta], S_T \rangle / S_T^0] \)
- History: Brennan and Schwartz (1976), Aase and Persson (1994), and many more
- Reasons: minimal martingale measure, quadratic approaches, utility approaches, Law of Large Numbers arguments
4 Life insurance contracts

- Benefits: $t$-portfolios $\gamma_t$
- premiums: $t$-portfolios $\delta_t$
- viewpoint of the insurer: company gets $\delta_t - \gamma_t$ at $t$

Important: How are premiums invested?

$\Rightarrow$ $\delta_r$ is seen together with the self-financing strategy $(\delta_{r,t})_{t \geq r}$ starting at $r$

<table>
<thead>
<tr>
<th>time</th>
<th>$r$</th>
<th>$r+1$</th>
<th>$r+2$</th>
<th>$\ldots$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>portfolio</td>
<td>$\delta_r = \delta_{r,r}$</td>
<td>$\delta_{r,r+1}$</td>
<td>$\delta_{r,r+2}$</td>
<td>$\ldots$</td>
<td>$\delta_{r,T}$</td>
</tr>
</tbody>
</table>

- benefits analogously
5 Market values and gains

- Market value of a life insurance contract at time $t$

$$MV_t = \sum_{r<t} \pi_t (\delta_{r,t} - \gamma_{r,t}) + \sum_{r\geq t} \pi_t (\delta_r - \gamma_r)$$  \hspace{2cm} (3)

- past stream

- future stream

- company's gain $G_{s,t}$ obtained during $[s, t]$

$$G_{s,t} := MV_t - MV_s$$  \hspace{2cm} (4)

- $MV_t$ and $G_{s,t}$ in $L^0(F \times B, \mathcal{F}_t \otimes \mathcal{B}_t, \mathbb{F} \otimes \mathbb{B})$
6 One-period decomposition

Orthogonal decomposition is proposed

\[
G_{s,t}^F = \mathbf{E}_{F \otimes B}[G_{s,t} | \mathcal{F}_t \otimes \mathcal{B}_s]
\]

(5)

\[
G_{s,t}^B = G_{s,t} - G_{s,t}^F
\]

(6)

“Natural” properties:

1. \(G_{s,t}^F \in L^0(F \times B, \mathcal{F}_t \otimes \mathcal{B}_s, \mathbb{F} \otimes \mathbb{B}) \Rightarrow G_{s,t}^F\) is replicable by a purely financial s.f. strategy starting at \(s\)

2. \(G_{s,t}^F\) closest to \(G_{s,t}\) w.r.t. \(\| \cdot \|_2\)

3. \(\mathbf{E}[G_{s,t}^B] = 0\)

4. biometric parts can be pooled (explained below)

5. for \(s = t - 1\), biometric parts do not(!) depend on trading strategy
7 Hedging

- For a random variable $Z$ in any $L^2(P, \mathcal{P}, \mathbb{P})$ its **conditional variance** w.r.t. a sub-$\sigma$-algebra $\mathcal{P}' \subset \mathcal{P}$ is defined by

$$ \text{Var}[X|\mathcal{P}'] = \mathbb{E}[(X - \mathbb{E}[X|\mathcal{P}'])^2|\mathcal{P}']. \quad (7) $$

- $p(s, t - s) =$ price of a ZCB with time to maturity $t - s$ at time $s$

**PROPOSITION 1 (Locally variance-optimal market value).** The locally variance-optimal market value at time $t$, which can be achieved by a purely financial s.f. strategy of cost 0 starting at $s$, is

$$ MV_{t}^{opt} = p(s, t - s)^{-1} MV_s + G_{s,t}^B. \quad (8) $$
Interpretation

- Minimization of fluctuation of $MV_t \Rightarrow MV_t^{\text{opt}}$ develops in the mean like riskless investment (seen from $s$)
- residual risk = biometric part of original gain
- cost of hedge = (-1)× cost of capital $MV_s$ at time $s$

Main ingredients of proof:

$$\Pi_s^t(G_{s,t}) = \Pi_s^t(G_{s,t}^F) = (1 - p(s, t - s))MV_s$$  \hspace{1cm} (9)
8 Pooling - a convergence property

- All considered independent individuals \( i \in \mathbb{N} \) will have a contract
- maximum life span \( \Delta \), maximum dates of death \( T_i \) \((T = \mathbb{N})\)
- bounded, possibly dependent portfolios; \( A^i_t := \{i \text{ signs at } t\} \)

**PROPOSITION 2.** Under the above assumptions,

\[
\frac{1}{m} \sum_{i=1}^{m} \sum_{t=0}^{T_i-1} 1_{A^i_t} \sum_{r=t+1}^{T_i} i G^B_{r-r-1,r}/S^0_r \xrightarrow{m \to \infty} 0 \quad \mathbb{F} \otimes \mathbb{B}-a.s. \quad (10)
\]

- The mean aggregated discounted biometric risk contribution per contract converges to zero a.s. for an increasing number of independent policyholders.
- independent from distribution of contracts on time axis!
- pooling (10) = core competence of insurance companies
COROLLARY 1. Assume that \((S^0_t)_{t \in \mathbb{N}}\) is the price process of the locally riskless money account and that the insurance company sells fairly priced contracts, only, i.e. \(1_{A_i}^i MV_t = 0\) for \(0 \leq t < T_i\) when \(^i MV_t\) denotes the present value (cf. (3)) of the \(i\)-th (meta-)contract at \(t\). Under the hedge of Proposition 1, started at the beginning of each (sub-)contract for each time period,

\[
\frac{1}{m} \sum_{i=1}^{m} \frac{i MV_{T_i}}{S^0_{T_i}} \xrightarrow{m \to \infty} 0 \quad \mathbb{P}\text{-a.s.} \tag{11}
\]

Interpretation. (11) is the mean discounted total gain (= discounted present value at \(T_i\)) of the first \(m\) contracts that converges to zero almost surely.
9 Conclusion

• Natural decomposition of risk by orthogonal projections
• pooling can/should be considered as a convergence property
• role of locally variance-optimal hedges explained
• Consequences for bonus theory?
References


http://www.ma.hw.ac.uk/~fischer/papers/valuation.pdf

http://www.ma.hw.ac.uk/~fischer/papers/decomp.pdf


