1 Proposition, Logical connectives and compound statements

Logic is the discipline that deals with the methods of reasoning. Intuitively speaking, logic as a subject is the collection of techniques used to prove that an argument is valid. The purpose of studying logic in computer science is to prove theorems and verify the correctness of programs. We would follow [1, 2, 3] for logic in this course.

We want to make statements which can be an atomic statement or a compound statement and determine whether the statement is true or false. In that sense, a statement is attached to a truth value.

A proposition (or a statement) is a declarative sentence that is either TRUE (denoted as T) or FALSE (denoted as F), but not both.

Exercise 1 Consider the statement $S : 3 - x = 5$. Is it a proposition?

2 Logical connectives and compound statements

Propositional variables are propositions usually denoted as $p, q, r, \ldots$. Propositional variables can be combined using logical connectives to obtain compound statements. The truth value of a compound statement depends only on the truth value of the statements and the logical connective.

Negation: The negation of $p$ is denoted as $\neg p$. $p$ is TRUE when $\neg p$ is FALSE. In terms of truth table, we have the following:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\neg p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Conjunction: The conjunction of $p$ and $q$ is denoted as $p \land q$. $p \lor q$ is true when both $p$ and $q$ are true.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \land q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Disjunction: The disjunction of $p$ and $q$ is denoted as $p \lor q$. $p \lor q$ is true when either $p$ or $q$ or both are true.
<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \lor q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
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<tr>
<td>F</td>
<td>F</td>
<td>F</td>
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</tbody>
</table>

**Exercise 2** Form the truth table for the compound statement \((p \land q) \lor (\neg p)\).

## 3 Quantifiers

Let us recall our definition of sets. We defined sets as the collection of all elements \( x \) satisfying a property \( P(x) \). In terms of set theoretic notations, \( \{ x \mid P(x) \} \). Consider an element \( w \) of this set. Surely, \( P(w) \) is true. Consider another element \( z \) not in this set. Surely, \( P(z) \) is false. \( P(x) \) is called to be a **predicate** or a **propositional function**. It is a propositional function because each choice of \( x \) produces a proposition \( P(x) \) that can either be true or false but not both.

**Example 1** Let \( A = \{ x \mid x \in \mathbb{R} \text{ and } x \leq \pi \} \). Here, the proposition is \( P(x) : \text{“}x \text{ is a real number less than equal to } \pi \text{”} \). \( P(1) \) is true, so \( 1 \in A \), but \( P(400) \) is false, so \( 400 \notin A \).

**Example 2** Consider the following piece of C code.

```c
unsigned int i, j;
    ....
    ....
while((i <= 5 ) && (j >= 6))
{
    printf("\n Hello World \n");
    i++; j--;
}
```

The statement \( i \leq 5 \) denotes a predicate or a propositional function as for some values of \( i \), the statement is true. Similarly, with \( j \geq 6 \). With the ‘&&’ sign, we make a conjunction of the two propositional functions to get a new predicate. This predicate acts as a control of how the while loop will run.

**Universal Quantification**: The universal quantification of a predicate \( P(x) \) is the statement: for all values of \( x \), \( P(x) \) is true, denoted as \( \forall x \ P(x) \). \( \forall \) is the universal quantifier.

**Example 3** Let a predicate \( P(x) \) be \( P(x) : x + x = 2x, x \in \mathbb{R} \). So, the universal quantification of \( P(x) \), \( \forall x P(x) \) is true.

**Existential Quantification**: The existential quantification of a predicate \( P(x) \) is the statement: there exists a value of \( x \) for which \( P(x) \) is true, denoted as \( \exists x P(x) \). \( \exists \) is the existential quantifier.
Example 4 Let a predicate $P(x) \equiv x^2 = 1, x \in \mathbb{R}$. So, the existential quantification of $P(x)$, $\exists x P(x)$ is true, as $P(1)$ is true. Consider the propositional function $P(y) : \exists y y + 2 = y$. $P(y)$ is false.

Example 5 This is an example of writing compound predicates. Let $P(x) : x$ is a person; $B(x, y) : x$ is the brother of $y$; $S(x, y) : x$ is the sister of $y$. We want to write the predicate: “$x$ is the sister of the brother of $y$”. In order to symbolize the predicate, we name a person called $z$ as the brother of $y$. So, the statement now is: there exists a person $z$ such that $x$ is the sister of $z$ AND $z$ is the brother of $y$. Thus, the predicate now is:

$$(\exists z)(P(z) \land S(x, z) \land B(z, y)).$$

Now, for a relation between existential quantifier and universal quantifier. Let $p$ be a predicate $\forall x P(x)$. The negation of $p$ is false when $p$ is true and true when $p$ is false. For $p$ to be false, there must be at least one value of $x$ for which $P(x)$ is false. Thus, $p$ is false if $\exists x, \neg P(x)$ is true. On the other hand, if $\exists x, \neg P(x)$ is false, then for every $x$, $\neg P(x)$ is false, i.e. $\forall x P(x)$ is true.

Exercise 3 Are the following two statements always equivalent: $\exists z \forall x \forall y p(x, y, z)$ and $\forall x \forall y \exists z p(x, y, z)$? Can you give an example of a $p(x, y, z)$ to show that the two statements are not equivalent?

4 Conditional statements

Let us first look at what are necessary and sufficient conditions.

Sufficient condition: We start with an example.

Example 6 Consider the following statement. If a natural number $n$ is even, then $n$ is divisible by 2. Look at the statement $p : a$ natural number $n$ is even, and the statement $q : n$ is divisible by 2. We say that $p$ is a sufficient condition for $q$.

Let $p$ and $q$ be two statements. We say that $p$ is a sufficient condition for $q$ when $p$ being true implies $q$ is true. Formally, a statement $p$ is a sufficient condition of a statement $q$ if $p$ implies ($\Rightarrow$) $q$. Another example. If you are studying M. Tech. (CS) at ISI then you have cleared the ISI entrance test. One more example. If $x$ is a rational number, then $x$ is real. In all the above examples, let $p$ be the first statement and $q$ be the second one. We can see that $p \Rightarrow q$. Let us explore a feature that is emerging from all the above examples. Let $S(p)$ be the set of objects for which $p$ holds true and $S(q)$ be the set of objects for which $q$ holds true. Then, $S(p) \subseteq S(q)$.
Necessary condition: We start with an example.

Example 7 Consider the following statement. If a natural number \( n > 2 \) is prime, then \( n \) is odd. Look at the statement \( p : \) \( n \) is odd, and the statement \( q : \) a natural number \( n \) greater than 2 is prime. We say that \( p \) is a necessary condition for \( q \).

A necessary condition of a statement must be satisfied for the statement to be true. Let \( p \) and \( q \) be two statements. \( p \) is a necessary condition of \( q \) if \( q \) implies (\( \Rightarrow \)) \( p \). The contrapositive of a statement \( q \Rightarrow p \) is \( \neg p \Rightarrow \neg q \). Then, we have another interpretation of a necessary statement: \( p \) is not true implies \( q \) is not true. As before, let \( S(p) \) be the set of objects for which \( p \) holds true and \( S(q) \) be the set of objects for which \( q \) holds true. Then, \( S(p) \supseteq S(q) \).

Necessary and sufficient condition: We say that a statement \( p \) is necessary and sufficient for \( q \) if \( p \) is necessary for \( q \) and \( p \) is sufficient for \( q \). We also say \( p \) if and only if \( q \) which is denoted as \( p \Leftrightarrow q \). So, to prove \( p \) is a necessary and sufficient condition for \( q \), we have to prove both \( p \Rightarrow q \) and \( q \Rightarrow p \).

We next look at truth tables of the compound statements corresponding to the conditional (\( \Rightarrow \)) and biconditional (\( \Leftrightarrow \)).

Conditional statement: Let \( p \) and \( q \) be statements. Then, the compound statement: if \( p \), then \( q \), denoted as \( p \Rightarrow q \) is called a conditional statement or an implication. \( p \) is said to be the hypothesis or antecedent and \( q \) is said to be the consequent or conclusion. “if ___ then ___” is a connective. The compound statement \( p \Rightarrow q \) is ‘true’ asserts the following: if \( p \) is ‘true’, then \( q \) will also be found to be ‘true’. \( p \Rightarrow q \) means \( q \) is a necessary condition for \( p \). The converse of the statement \( p \Rightarrow q \) is \( q \Rightarrow p \). Let us look at the truth table:

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \Rightarrow q )</th>
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<tbody>
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Exercise 4 Using truth table, verify that \( p \Rightarrow q \) and \( \neg q \Rightarrow \neg p \) are the same implications.

Biconditional statement: Let \( p \) and \( q \) be statements. Then, the compound statement: \( p \) if and only if \( q \), denoted as \( p \Leftrightarrow q \) is called a biconditional statement or an equivalence. The compound statement \( p \Leftrightarrow q \) is ‘true’ only when both \( p \) and \( q \) are true or when both \( p \) and \( q \) are false. \( p \Leftrightarrow q \) means \( p \) is a necessary and sufficient condition for \( q \). Let us look at the truth table:
Exercise 5  Compute the truth table for \((p \Rightarrow q) \iff (\neg q \Rightarrow \neg p)\).

Tautology and contradiction A statement that is true for all possible values of its propositional variables is called a tautology. A statement that is always false is called a contradiction. \(p\) and \(q\) are logically equivalent, denoted as \(\equiv\), if \(p \iff q\) is a tautology.

Exercise 6  Prove the commutative, associative, distributive, idempotent properties and De Morgan’s law using the notion of equivalence.

Exercise 7  Show that
- \(((p \Rightarrow r) \land (q \Rightarrow r)) \Rightarrow ((p \lor q) \Rightarrow r)\) is a tautology.
- \((p \Rightarrow q) \equiv (\neg p \lor q)\)
- \((p \Rightarrow q) \equiv (\neg q \Rightarrow \neg p)\)
- \((p \iff q) \equiv (p \Rightarrow q) \land (q \Rightarrow p)\)
- \(\neg(p \Rightarrow q) \equiv (p \land \neg q)\)
- \(\neg(p \iff q) \equiv ((p \land \neg q) \lor (q \land \neg p))\)
- \(\neg(\forall x \ p(x)) \equiv (\exists x \ (\neg p(x)))\)
- \(\neg(\exists x \ p(x)) \equiv (\forall x \ (\neg p(x)))\)

References