The grades K–4 Curriculum and Evaluation Standards for School Mathematics (NCTM 1989) encourages teachers to broaden and develop their students' mathematical understandings by providing students with opportunities to explore and discuss patterns and relationships. A crucial aspect of this view is for students to realize that the role of the variable extends beyond that of an “unknown,” particularly the role variables play in generalizing patterns and in describing change relationships among quantities (Philipp 1992; Usiskin 1988). The notion that young students can experience on an informal basis relationships describing change is now realized as important to the development of algebraic reasoning in early grades as recommended by reform documents.

With these thoughts in mind and using literature as our vehicle for developing algebraic reasoning, we taught in two classrooms to better determine how algebra can be informally introduced into the grades K–4 curriculum. In this article we share our experiences and discuss what we learned about the manner in which students describe and understand relationships involving change. In each case, the teacher to whom we refer is one of us. The primary focus of our instruction was to provide opportunities for students to facilitate the development of their algebraic reasoning so as to be better prepared to study formal algebra. We selected two books and two grade levels to assist us with our endeavors. The Doorbell Rang (Hutchins 1986) was used for a first-grade class, and One Hundred Hungry Ants (Pinczes 1993) was selected for a fourth-grade class.

The Doorbell Rang in First Grade

As the story unfolds in The Doorbell Rang (see fig. 1), two children are equally sharing twelve cookies that mom just baked, and then the doorbell rings. Two more children enter to share the cookies fairly, and then again the doorbell rings and two more children arrive who will also share the twelve cookies. As the story unfolds, it is apparent that as the number of children increases, the number of cookies each child receives decreases. The mathematical idea of change, as described in this situation, is an inverse relationship between two...
variables, that is, as one quantity increases, another quantity decreases, extending the role of variable beyond that of being an “unknown.”

After reading the story, a two-column chart (see fig. 2) was carefully developed on the chalkboard as the students discussed what happened as more children arrived. We found it important in developing algebraic reasoning to have the students describe the situation and explain their answers and to realize that the number of cookies each child receives depends on the number of children. The students were quick to realize this fact.

We asked the students to describe what they saw happening with the two columns of numbers. They responded that as the number of children got larger, the number of cookies each received got smaller, and they were able to describe their reasoning as to why this result was occurring. For example, John said, “Because there’s more people, more people come and the cookies go down less because you have to share them evenly.”

Although the situation in the book went from two children with six cookies each to twelve children with one cookie each, we added the additional number of 24 (referring to the number of children) to the chart to ascertain whether the students could extend their quantitative reasoning about change. The discussion proceeded as follows:

**Teacher:** We’re going to pretend that twenty-four kids come in the door. What happens to the number of cookies? We’ve already decided it’s going down because the more that come in, the numbers on the other side [of the chart] go down. If when there were twelve, they each got one, what happens when there’s twenty-four children?

**Anthony:** They each get one-half.

**Teacher:** I’m going to write “one-half” up here [on the chart]. If you have twelve of these cookies and 24 children, how many halves will you have?

**Colton:** Twenty-four.

**Teacher:** Emily, do you agree with that? [Emily agrees.] So, why do you think I picked the number 24? I went from 6 to 12 to 24. Why did I do that?

**Brian:** Maybe because it’s 24 and then you took 12 away.

**Teacher:** I see what you’re doing. I went from 6 to 12. Why did I go from 6 to 12?

**Megan:** Because you added on 6.

As illustrated, the first graders initially focused only on additive relationships by responding “you took 12 away” or “you added on 6.” Since we wanted to focus on the multiplicative relationship between the numbers, we continued to probe the relationship of the numbers as follows:

**Teacher:** One of the things … that Brian said is that he added on. What did I do going from here to here [6 to 12] and then from here to here [12 to 24] besides adding on? Katie?

**Katie:** It’s 6 and 6 is 12, 12 and 12 is 24.

**Teacher:** Okay, what else am I doing? John?

**John:** Pairing.

**Teacher:** Maria, what else am I doing?

**Maria:** You’re doubling.

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<table>
<thead>
<tr>
<th>Number of Children</th>
<th>Number of Cookies</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
</tr>
<tr>
<td>24</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 2. The students produced a chart to track cookies and children.
Teacher: I doubled it. That’s another thing I did, too, besides pairing, John. Very good. What does it mean to double something?

Maria: If you double a number, it means that if you get that number twice, it goes to a higher number.

Teacher: So, I doubled here [referring to the number of people on the chart], and I doubled here. When I doubled the number of people, let’s look at what happened to the number of cookies.

John: If more people came, they have to split in half the cookies still.

This discussion on doubling proceeded reasonably easily, and many children continued to explain their answers using the phrase “splitting in half.”

Pictorial representations helped students to focus on their reasoning.

We found that using pictorial representations helped the students to focus on their reasoning rather than on what operation to use with the numbers. It was important to relate the children’s explanations of doubling with a pictorial representation. At one point, the teacher drew on the chalkboard the situation of two children with the number of cookies each had (see fig. 3).

Again, we asked the students to describe what happens to the number of cookies each child receives when the number of children doubles from two to four. The students could see from the chart that each child would then receive three cookies. However, the teacher probed for the reason by drawing the students’ attention to the pictorial representation on the chalkboard, wherein two additional children had been added. She said that if “I doubled the amount of people, what happens to the [number of] cookies?” Aaron was able to show his reasoning of “splitting in half” by illustrating on the chalkboard how the two groups of six cookies could be split into four groups of three cookies by splitting a group of six into two groups of three.

Teacher: First of all, we have two people [referring to the picture on the chalkboard]; each has six cookies. Now, if two more people come, so that I doubled the number of people, what happens to the cookies? Aaron, what would you do?

Aaron: They each get three cookies.

Teacher: Show us how you would do that.

Aaron: These three here to there and these three. (See fig. 4.)

Teacher: When the number of people doubled, the number of cookies for this person … what had to happen, Maria?

Maria: They had to split it … in half.

In the case of how many cookies each child receives when there are forty-eight children, the students gave answers of “one-fourth” or “one-half of one-half.”

Fig. 3. Two children each with six cookies is illustrated.

Fig. 4. Four children are pictured, having three cookies each.
Teacher: What would that new number be if I doubled that number? [24] Kramer?

Kramer: 48.

Teacher: I just doubled the number of people. What does that tell you about the number of cookies that each person’s going to get?

Maria: I think you have to split it into sections. The cookies.

Teacher: Well, how were they split with twenty-four?

Maria: In half.

Teacher: Can you come and show me?

Maria represented the cookies being split in half and then in half again for the forty-eight people (see fig. 5).

Fig. 5. The cookie is divided into fourths.

To provide a generalization focusing on the multiplicative relationship, the teacher then asked, “When the number of children doubles, what happens to the number of cookies each child receives?” Some students realized that as the number of children doubles, the groups of cookies will always be “split in half,” the term they used. So the students realized that the number of cookies each child would receive would be only half the previous number. Thus in this way students were experiencing the mathematical concept of change as described by an inverse relationship between two quantities.

An extension of this lesson was to focus on what happens to the number of cookies each child receives if the number of children is tripled when the number of children is increased from 2 to 6.

Teacher: Look what I’m going to do to these numbers. We have doubled here and the number of cookies split in half. What happens if I go from two cookies to six cookies. It didn’t double. What did I do?

Theodore: You added four more on.

Teacher: That’s one thing that happened. What’s another word [that describes what happened] besides adding?

John: You tripled…. You did it three times.

Teacher: Show me on the board. Start with two people. Now, what did I do with the number of people?

John: Tripled.

Teacher: How many people is he going to have if he triples the number? Anna?

Anna: Six … because 2 plus 2 plus 2 is 6.

Teacher: Now what happens to these cookies that these two people have if we triple the number of people? Emily, show us what’s going to happen to the twelve we have.

Emily goes to the chalkboard, points to the cookies, and says, “Two will go here, and these two will go here, and then these two will go here, and these two will go here, and these two will go here, and these two will go here.” The teacher asked John to use Emily’s explanation and draw circles around groups of cookies to show what children get which cookies (see fig. 6).

Fig. 6. The cookies are divided into groups.

What happens if the number of children is tripled?

The idea of change as described by an inverse relationship between the variables (the number of children and the number of cookies each child receives) was the primary focus of this lesson. As we worked, we found it important to encourage students to describe what is happening and to provide explanations for their responses. It appeared that a crucial part of these explanations was the pictorial representations. We found that if we proceeded too quickly to “make a chart,” the focus seemed to shift to numerical calculating and
away from reasoning. Thus, “Why?” was an important question to keep asking. Also it was important to ask “What does the 2 [the 4, the 6, or the 12] represent?” and “What does the 6 [the 3, the 2, or the 1] represent?” This questioning helped to keep the focus on the quantities involved and their relationships rather than just on the numbers and the numerical relationships. The context of the situation was also important to developing the understanding that a change in one quantity caused a change in a second quantity. Further, if the context was not kept in focus, the emphasis of the students seemed to shift away from understanding the relationships of the changes to understanding just the properties of numbers and computation.

Students should describe what is happening and explain their responses.

We suggest that during the school year, the idea of doubling and splitting in half or tripling and splitting in thirds can be developed as early as first grade. Over time the students would become more flexible in their thinking, that is, they would not only be focusing on adding on or taking away, which is typical of the curriculum content in first grade. This flexibility in thought, we believe, is necessary to be able to reason algebraically and can be successfully developed at the primary level.

One Hundred Hungry Ants in Fourth Grade

The story of One Hundred Hungry Ants is about 100 ants going to a picnic in one long line (see fig. 7). Early in the story, the ants thought that they could all get to the picnic faster if they formed two lines instead of one, even faster if they formed four lines, and faster yet if they formed five lines. Ultimately they arrived at the picnic late after all the time spent reorganizing themselves in lines of ten. The mathematical idea of change found in this story is similar to that found in the story read to the first-grade class.

Working with fourth graders, we started in much the same way as we did in the first grade, making a chart on the chalkboard with the students quickly providing the numbers in the second column (see fig. 8).

Students were then asked to describe the relationships they saw. One student said that “as the number of rows get larger, the number of ants in a row gets smaller.” In fact, most students realized that as the number of lines increased, the number of ants in each line decreased. Further, the students recognized other relationships, such as when the numbers across from each other on the chart are multiplied, they always produce an answer of 100. When asked to fill in other possible numbers for the chart, the students added the following:

<table>
<thead>
<tr>
<th>Number of Lines</th>
<th>Number of Ants in Each Line</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>5</td>
</tr>
<tr>
<td>25</td>
<td>4</td>
</tr>
<tr>
<td>50</td>
<td>2</td>
</tr>
</tbody>
</table>

The teacher asked, “What if I double the number of lines. What happens to the number of ants in each line?” One student responded, “Divided by 2.” A discussion ensued that provided examples of where on the chart this relationship occurs.

At this point we developed the concern that the focus of the lesson had become one of computation. That is, the students knew that division is the inverse of multiplication and that multiplying the number of lines by 2 meant dividing the number of ants in each line by 2. The students were focusing on the multiplying and dividing aspects of the numbers in the chart rather than on the reasons for the changes in the numbers of ants and lines. The
first graders had not done so, probably because of their limited experiences with multiplication and division facts. To bring the focus back on quantitative reasoning, we modified our teaching strategy by reinforcing the importance of having the students develop a strategy that would lend itself to a reasoning process about changes rather than focus on the computation. We believe that this strategy ensued because the fourth graders quickly recognized the number facts of the numbers being used.

To shift our focus back to our purpose of developing quantitative reasoning, we used pictorial representations. The teacher drew a line of points on the chalkboard and suggested that it represented the 100 ants. Then she asked, “If the number of lines is one, what happens to the number of ants in each line when we double the number of lines?” One student said, “It is divided into half and half the ants,” meaning that 100 is divided or split in half and that 50 ants are in each half. When the teacher asked what happens when the number of lines doubled again, the students realized that each group of 50 ants is split into half and that the number of lines would be four and the number of ants in each line would be half of 50, or 25. We illustrated this fact by using the pictorial representation of what was happening to the number of lines. Students were also able to explain what happened to the number of ants in a line if the number of lines of ants was five times as many. The teacher asked if it were possible to have eight lines, and the students explained why some numbers could not be used if we were talking about whole ants!

As an extension, we changed our context to cookies (because cookies can be split in half) in a line to be able to develop our chart further and to focus on reasoning about quantities.

**Teacher:** We're going from twenty rows to forty. We're doubling the number of rows. What happens to the number of cookies in a row? It used to be five; it's now going to be what?

**Marcia:** Two and a half.

**Teacher:** Two and a half. Why?

**Sergio:** There's two cookies and then 2 times 2 is 4 and that's only part of 5, and then you divide the other cookie in half.

We believe that students' focusing on the kind of changes that can take place between two quantities is important in developing their preparation for algebra.
Reflections

In high school algebra courses, the type of change relationship described in our two stories is referred to as inverse variation and described algebraically as $y = \frac{k}{x}$. Even though we did not use this algebraic representation, the primary-age students were still able to discuss and understand the concept, since we focused on quantities and their relationships and not just on the numbers and the numerical relationships. Although our emphasis was on change described by an inverse relationship, both situations are also examples of functions because a value for one quantity determines a value for the other quantity. While developing the charts, we focused on “the number of” to reinforce the concept that a variable represents a number and is not merely a label for children, cookies, ants, or lines. The use of a letter as a label seems to present students with difficulties when first studying algebra.

Since pictorial representations seemed to assist the students in developing an understanding of how a change in one quantity produced a change in a second, we prefer that students draw their own pictures because it is easy for teachers to misrepresent the reasoning of the student. Further, we believe that even young students can reason algebraically if given the opportunity. However, we also believe that unless these opportunities are provided, understanding formal algebra will continue to be a struggle for most students. We suggest that teachers’ instructional decision making incorporate mathematics topics that encourage flexibility of thought about situations involving patterns and relationships among quantities. To do so successfully, the focus of instruction needs to be on reasoning, not on the manipulation of numbers. We found it helpful to be aware of recommendations in such reform documents as the *Curriculum and Evaluation Standards*, which provided the basis for our decision making about the type of reasoning opportunities that we could give primary-age students.

We believe that opportunities exist for primary-age students to experience algebraic reasoning with patterns and relationships and that these situations should be varied to include multiplicative reasoning and inverse variation. Children’s literature was an effective vehicle for us to use to explore change and relationships and begin to develop algebraic reasoning.

References


