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**Big Ideas Math**

**Integrated Mathematics**

**Sampler**

**Ron Larson**

**Laurie Boswell**
Big Ideas Learning is pleased to introduce a new high school program, *Big Ideas Math Integrated Mathematics I, II, and III*. The program was written by renowned authors Ron Larson and Laurie Boswell and was developed using the consistent, dependable learning and instructional theory that have become synonymous with *Big Ideas Math*. Students will gain a deeper understanding of mathematics by narrowing their focus to fewer topics at each grade level. They will also master content through inductive reasoning opportunities, engaging explorations, concise stepped-out examples, and rich thought-provoking exercises.

The *Big Ideas Math Integrated Mathematics* research-based curriculum features a continual development of concepts that have been previously taught while integrating algebra, geometry, probability, and statistics topics throughout each course.

In *Integrated Mathematics I*, students will study linear and exponential equations and functions. Students will use linear regression and perform data analysis. They will also learn about geometry topics such as simple proofs, congruence, and transformations.

*Integrated Mathematics II* expands into quadratic, absolute value, and other functions. Students will also explore polynomial equations and factoring, and probability and its applications. Coverage of geometry topics extends to polygon relationships, proofs, similarity, trigonometry, circles, and three-dimensional figures.

In *Integrated Mathematics III*, students will expand their understanding of area and volume with geometric modeling, which students will apply throughout the course as they learn new types of functions. Students will study polynomial, radical, logarithmic, rational, and trigonometric functions. They will also learn how visual displays and statistics relate to different types of data and probability distributions.
About the Authors

Ron Larson, Ph.D., is well known as the lead author of a comprehensive program for mathematics that spans middle school, high school, and college courses. He holds the distinction of Professor Emeritus from Penn State Erie, The Behrend College, where he taught for nearly 40 years. He received his Ph.D. in mathematics from the University of Colorado. Dr. Larson’s numerous professional activities keep him actively involved in the mathematics education community and allow him to fully understand the needs of students, teachers, supervisors, and administrators.

Laurie Boswell, Ed.D., is the Head of School and a mathematics teacher at the Riverside School in Lyndonville, Vermont. Dr. Boswell is a recipient of the Presidential Award for Excellence in Mathematics Teaching and has taught mathematics to students at all levels, from elementary through college. Dr. Boswell was a Tandy Technology Scholar and served on the NCTM Board of Directors from 2002 to 2005. She currently serves on the board of NCSM and is a popular national speaker.

Dr. Ron Larson and Dr. Laurie Boswell began writing together in 1992. Since that time, they have authored over two dozen textbooks. In their collaboration, Ron is primarily responsible for the student edition while Laurie is primarily responsible for the teaching edition.
Program Resources

Print

Student Edition
- Designed to the UDL Guidelines

Teaching Edition
- Laurie's Notes

Student Journal
Available in English and Spanish

Resources by Chapter
- Start Thinking
- Warm Up
- Cumulative Review Warm Up
- Practice A and B
- Enrichment and Extension
- Puzzle Time
- Family Communication Letters
  Available in English and Spanish

Assessment Book
- Performance Tasks
- Prerequisite Skills Test with Item Analysis
- Quarterly Standards Based Tests
- Quizzes
- Chapter Tests
- Alternative Assessments with Scoring Rubrics
- Pre-Course Test with Item Analysis
- Post Course Test with Item Analysis

Technology

Student Edition
With complete English and Spanish audio
- Dynamic eBook App
- Dynamic Solutions Tool
- Dynamic Investigations
- Lesson Tutorial Videos

Dynamic Classroom
- Vocabulary Flash Cards
- Worked-Out Solutions
- Extra Examples
- Warm Up and Closure Activities

Dynamic Teaching Tools
- Interactive Whiteboard Lesson Library
  • Compatible with SMART®, Promethean®, and Mimio® technology
- Real-Life STEM Videos
- Editable Online Resources
  • Lesson Plans
  • Assessment Book
  • Resources by Chapter
  • Differentiating the Lesson
- Answer Presentation Tool

Dynamic Assessment and Progress Monitoring Tool
- Assessment Creation and Delivery
- Progress Monitoring

Multilingual Glossary
- Key mathematical vocabulary terms in 14 languages
Program Overview

Program Philosophy: Rigor and Balance with Real-Life Applications

The Big Ideas Math® program balances conceptual understanding with procedural fluency. Real-life applications help turn mathematical learning into an engaging and meaningful way to see and explore the real world.

**Essential Question**

How can you factor a polynomial?

**Exploration 1**

**Factoring Polynomials**

**Work with a partner.** Match each polynomial equation with the graph of its related polynomial function. Use the x-intercepts of the graph to write each polynomial in factored form. Explain your reasoning.

- a. \( x^2 + 5x + 4 = 0 \)
- b. \( x^3 - 2x^2 - x + 2 = 0 \)
- c. \( x^3 + x^2 - 2x = 0 \)
- d. \( x^3 - x = 0 \)
- e. \( a^4 - 5a^2 + 4 = 0 \)
- f. \( x^3 - 2x^3 - 2x = 0 \)

**Example 1**

**Finding a Common Monomial Factor**

Factor each polynomial completely.

- a. \( x^3 - 4x^2 - 5x \)
- b. \( 3y^3 - 48y^3 \)
- c. \( 5x^3 + 30x^2 + 45x^2 \)

**Solution**

- a. \( x^3 - 4x^2 - 5x = x(x^2 - 4x - 5) \)
  \( = x(x - 5)(x + 1) \)
- b. \( 3y^3 - 48y^3 = 3y^3(y^2 - 16) \)
  \( = 3y^3(y - 4)(y + 4) \)
- c. \( 5x^3 + 30x^2 + 45x^2 = 5x^2(x^2 + 6x + 9) \)

**Example 7**

**Real-Life Application**

During the first 5 seconds of a roller coaster ride, the function \( h(t) = 4t^3 - 21t^2 + 9t + 34 \) represents the height \( h \) (in feet) of the roller coaster after \( t \) seconds.

How long is the roller coaster at or below ground level in the first 5 seconds?

**Solution**

1. **Understand the Problem**
   You are given a function rule that represents the height of a roller coaster. You are asked to determine how long the roller coaster is at or below ground during the first 5 seconds of the ride.

2. **Make a Plan**
   Use a graph to estimate the zeros of the function and check using the Factor Theorem. Then use the zeros to describe where the graph lies below the \( x \)-axis.

3. **Solve the Problem**
   From the graph, two of the zeros appear to be \( -1 \) and 2. The third zero is between 4 and 5.
Dynamic Technology Package

The Big Ideas Math program includes a comprehensive technology package that enhances the curriculum and allows students to engage with the underlying mathematics in the text.

Dynamic Student Edition
The Dynamic Student Edition gives students access to the complete textbook and robust embedded digital resources. Interactive investigations, direct links to remediation, and additional resources are linked at point-of-use. Students can customize their Dynamic Student Editions through note taking and bookmarking, and it can also be accessed offline after it has been downloaded to a device. Audio support and Lesson Tutorial Videos are also included in English and Spanish.

Dynamic Investigations
Dynamic Investigations are powered by Desmos® and GeoGebra®. These interactivities expand on the Explorations in the program and allow students to learn mathematics through a hands-on approach. Teachers and students can integrate these investigations into their discovery learning.

Real-Life STEM Videos
Every chapter in the program contains a Real-Life STEM Video that ties directly to a Performance Task. Students can explore topics like the speed of light, natural disasters, and wind power while applying their knowledge to a comprehensive project or task.

Dynamic Classroom
The Dynamic Classroom is an online interactive version of the Student Edition that can be used as a lesson presentation tool. Teachers can present their lessons and have point-of-use access to all of the online resources available that supplement every section of the program.

Dynamic Teaching Tools
These tools include an Interactive Whiteboard Lesson Library that includes customizable lessons and templates for every section in the program. Lessons are compatible with SMART®, Promethean®, and Mimio® whiteboards. The Answer Presentation Tool can be used to display worked-out solutions to homework and test problems from Big Ideas Math content.

Dynamic Assessment and Progress Monitoring Tool
This online tool allows teachers to create tests by standard or by Big Ideas Math content. Teachers can assign any exercise from the student textbook, problems from the program’s ancillary pieces, or additional items from the item bank, and students can complete their assignments within the tool’s interface.

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3.2 Adding, Subtracting, and Multiplying Polynomials
3.3 Dividing Polynomials
3.4 Factoring Polynomials
3.5 Solving Polynomial Equations
3.6 The Fundamental Theorem of Algebra
3.7 Transformations of Polynomial Functions
3.8 Analyzing Graphs of Polynomial Functions
3.9 Modeling with Polynomial Functions

SEE the Big Idea

Basketball (p. 134)
Ruins of Caesarea (p. 151)
Quonset Hut (p. 174)
Electric Vehicles (p. 115)
Zebra Mussels (p. 159)
Maintaining Mathematical Proficiency

Simplifying Algebraic Expressions (Grade 7)

Example 1 Simplify the expression $9x + 4x$.

$$9x + 4x = (9 + 4)x$$
$$= 13x$$

Distributive Property
Add coefficients.

Example 2 Simplify the expression $2(x + 4) + 3(6 - x)$.

$$2(x + 4) + 3(6 - x) = 2x + 2(4) + 3(6) + 3(-x)$$
$$= 2x + 8 + 18 - 3x$$
$$= 2x - 3x + 8 + 18$$
$$= -x + 26$$

Distributive Property
Multiply.
Group like terms.
Combine like terms.

Simplify the expression.

1. $6x - 4x$
2. $12m - m - 7m + 3$
3. $3(y + 2) - 4y$
4. $9x - 4(2x - 1)$
5. $-(z + 2) - 2(1 - z)$
6. $-x^2 + 5x + x^2$

Solving Quadratic Equations by Factoring (Math II)

Example 3 Solve $x^2 + 7x = 18$.

$$x^2 + 7x = 18$$ Write equation.
$$x^2 + 7x - 18 = 0$$ Subtract 18 from each side.
$$(x + 9)(x - 2) = 0$$ Factor left side.
$$x + 9 = 0$$ or $$x - 2 = 0$$ Zero-Product Property
$$x = -9$$ or $$x = 2$$ Solve for $x$.

The solutions are $x = -9$ and $x = 2$.

Solve the equation by factoring.

7. $x^2 + 3x + 2 = 0$
8. $x^2 - 6x + 8 = 0$
9. $x^2 + 10x = -25$
10. $2x^2 - 84 = 2x$
11. $4x^2 = 12x - 9$
12. $8x - 3 = -3x^2$

13. **ABSTRACT REASONING** Explain how you can find the solutions of an equation of the form $(x - a)(x - b)(x - c) = 0$. 

Dynamic Solutions available at BigIdeasMath.com

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Using Technology to Explore Concepts

Continuous Functions
A function is **continuous** when its graph has no breaks, holes, or gaps.

**Core Concept**

**Mathematical Practices**
Mathematically proficient students use technological tools to explore concepts.

**EXAMPLE 1** Determining Whether Functions Are Continuous

Use a graphing calculator to compare the two functions. What can you conclude? Which function is not continuous?

\[ f(x) = x^2 \quad \text{and} \quad g(x) = \frac{x^3 - x^2}{x - 1} \]

**SOLUTION**

The graphs appear to be identical, but \( g \) is not defined when \( x = 1 \). There is a **hole** in the graph of \( g \) at the point \( (1, 1) \). Using the **table** feature of a graphing calculator, you obtain an error for \( g(x) \) when \( x = 1 \). So, \( g \) is not continuous.

**Monitoring Progress**
Use a graphing calculator to determine whether the function is continuous. Explain your reasoning.

1. \( f(x) = \frac{x^2 - x}{x} \)
2. \( f(x) = x^3 - 3 \)
3. \( f(x) = \sqrt{x^2 + 1} \)
4. \( f(x) = |x + 2| \)
5. \( f(x) = \frac{1}{x} \)
6. \( f(x) = \frac{1}{\sqrt{x^2 - 1}} \)
7. \( f(x) = x \)
8. \( f(x) = 2x - 3 \)
9. \( f(x) = \frac{x}{x} \)
3.1 Graphing Polynomial Functions

**Essential Question**  What are some common characteristics of the graphs of cubic and quartic polynomial functions?

A polynomial function of the form

\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \]

where \( a_n \neq 0 \), is cubic when \( n = 3 \) and quartic when \( n = 4 \).

**EXPLORATION 1** Identifying Graphs of Polynomial Functions

*Work with a partner.* Match each polynomial function with its graph. Explain your reasoning. Use a graphing calculator to verify your answers.

a. \( f(x) = x^3 - x \)  
   b. \( f(x) = -x^3 + x \)  
   c. \( f(x) = -x^4 + 1 \)  
   d. \( f(x) = x^4 \)  
   e. \( f(x) = x^3 \)  
   f. \( f(x) = x^4 - x^2 \)

**EXPLORATION 2** Identifying x-Intercepts of Polynomial Graphs

*Work with a partner.* Each of the polynomial graphs in Exploration 1 has x-intercept(s) of \(-1, 0, \) or \(1\). Identify the x-intercept(s) of each graph. Explain how you can verify your answers.

**Communicate Your Answer**

3. What are some common characteristics of the graphs of cubic and quartic polynomial functions?

4. Determine whether each statement is *true* or *false*. Justify your answer.
   a. When the graph of a cubic polynomial function rises to the left, it falls to the right.
   b. When the graph of a quartic polynomial function falls to the left, it rises to the right.
### 3.1 Lesson

**Core Vocabulary**
- polynomial, p. 112
- polynomial function, p. 112
- end behavior, p. 113

**Previous**
- monomial
- linear function
- quadratic function

---

### What You Will Learn

- Identify polynomial functions.
- Graph polynomial functions using tables and end behavior.

---

### Polynomial Functions

Recall that a monomial is a number, a variable, or the product of a number and one or more variables with whole number exponents. A **polynomial** is a monomial or a sum of monomials. A **polynomial function** is a function of the form

\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \]

where \( a_n \neq 0 \), the exponents are all whole numbers, and the coefficients are all real numbers. For this function, \( a_n \) is the **leading coefficient**, \( n \) is the **degree**, and \( a_0 \) is the **constant term**. A polynomial function is in **standard form** when its terms are written in descending order of exponents from left to right.

You are already familiar with some types of polynomial functions, such as linear and quadratic. Here is a summary of common types of polynomial functions.

<table>
<thead>
<tr>
<th>Degree</th>
<th>Type</th>
<th>Standard Form</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Constant</td>
<td>( f(x) = a_0 )</td>
<td>( f(x) = -14 )</td>
</tr>
<tr>
<td>1</td>
<td>Linear</td>
<td>( f(x) = a_1 x + a_0 )</td>
<td>( f(x) = 5x - 7 )</td>
</tr>
<tr>
<td>2</td>
<td>Quadratic</td>
<td>( f(x) = a_2 x^2 + a_1 x + a_0 )</td>
<td>( f(x) = 2x^2 + x - 9 )</td>
</tr>
<tr>
<td>3</td>
<td>Cubic</td>
<td>( f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0 )</td>
<td>( f(x) = x^3 - x^2 + 3x )</td>
</tr>
<tr>
<td>4</td>
<td>Quartic</td>
<td>( f(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 )</td>
<td>( f(x) = x^4 + 2x - 1 )</td>
</tr>
</tbody>
</table>

---

### Example 1

**Identifying Polynomial Functions**

Decide whether each function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

- **a.** \( f(x) = -2x^3 + 5x + 8 \)
- **b.** \( g(x) = -0.8x^3 + \sqrt{2}x^4 - 12 \)
- **c.** \( h(x) = -x^2 + 7x^{-1} + 4x \)
- **d.** \( k(x) = x^2 + 3x \)

**SOLUTION**

- **a.** The function is a polynomial function that is already written in standard form. It has degree 3 (cubic) and a leading coefficient of \(-2\).
- **b.** The function is a polynomial function written as \( g(x) = \sqrt{2}x^4 - 0.8x^3 - 12 \) in standard form. It has degree 4 (quartic) and a leading coefficient of \( \sqrt{2} \).
- **c.** The function is not a polynomial function because the term \( 7x^{-1} \) has an exponent that is not a whole number.
- **d.** The function is not a polynomial function because the term \( 3x \) does not have a variable base and an exponent that is a whole number.

---

### Monitoring Progress

Decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

1. \( f(x) = 7 - 1.6x^2 - 5x \)
2. \( p(x) = x + 2x^{-2} + 9.5 \)
3. \( q(x) = x^3 - 6x + 3x^4 \)
EXAMPLE 2  Evaluating a Polynomial Function

Evaluate \( f(x) = 2x^4 - 8x^2 + 5x - 7 \) when \( x = 3 \).

**SOLUTION**

\[
\begin{align*}
f(x) &= 2x^4 - 8x^2 + 5x - 7 \\
f(3) &= 2(3)^4 - 8(3)^2 + 5(3) - 7 \\
&= 162 - 72 + 15 - 7 \\
&= 98
\end{align*}
\]

The **end behavior** of a function's graph is the behavior of the graph as \( x \) approaches positive infinity (\( +\infty \)) or negative infinity (\( -\infty \)). For the graph of a polynomial function, the end behavior is determined by the function’s degree and the sign of its leading coefficient.

Core Concept

End Behavior of Polynomial Functions

<table>
<thead>
<tr>
<th>Degree: odd</th>
<th>Leading coefficient: positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) \rightarrow +\infty ) as ( x \rightarrow +\infty )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Degree: odd</th>
<th>Leading coefficient: negative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) \rightarrow -\infty ) as ( x \rightarrow +\infty )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Degree: even</th>
<th>Leading coefficient: positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) \rightarrow +\infty ) as ( x \rightarrow -\infty )</td>
<td></td>
</tr>
<tr>
<td>( f(x) \rightarrow +\infty ) as ( x \rightarrow +\infty )</td>
<td></td>
</tr>
</tbody>
</table>

EXAMPLE 3  Describing End Behavior

Describe the end behavior of the graph of \( f(x) = -0.5x^4 + 2.5x^2 + x - 1 \).

**SOLUTION**

The function has degree 4 and leading coefficient \(-0.5\). Because the degree is even and the leading coefficient is negative, \( f(x) \rightarrow -\infty \) as \( x \rightarrow -\infty \) and \( f(x) \rightarrow -\infty \) as \( x \rightarrow +\infty \). Check this by graphing the function on a graphing calculator, as shown.

Check

Evaluate the function for the given value of \( x \).

4. \( f(x) = -x^3 + 3x^2 + 9; \ x = 4 \)
5. \( f(x) = 3x^5 - x^4 - 6x + 10; \ x = -2 \)
6. Describe the end behavior of the graph of \( f(x) = 0.25x^3 - x^2 - 1 \).
Graphing Polynomial Functions

To graph a polynomial function, first plot points to determine the shape of the graph's middle portion. Then connect the points with a smooth continuous curve and use what you know about end behavior to sketch the graph.

**Example 4** Graphing Polynomial Functions

Graph (a) \( f(x) = -x^3 + x^2 + 3x - 3 \) and (b) \( f(x) = x^4 - x^3 - 4x^2 + 4 \).

**SOLUTION**

a. To graph the function, make a table of values and plot the corresponding points. Connect the points with a smooth curve and check the end behavior.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>3</td>
<td>-4</td>
<td>-3</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

The degree is odd and the leading coefficient is negative. So, \( f(x) \to +\infty \) as \( x \to -\infty \) and \( f(x) \to -\infty \) as \( x \to +\infty \).

b. To graph the function, make a table of values and plot the corresponding points. Connect the points with a smooth curve and check the end behavior.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>12</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>-4</td>
</tr>
</tbody>
</table>

The degree is even and the leading coefficient is positive. So, \( f(x) \to +\infty \) as \( x \to -\infty \) and \( f(x) \to +\infty \) as \( x \to +\infty \).

**Example 5** Sketching a Graph

Sketch a graph of the polynomial function \( f \) having these characteristics.

- \( f \) is increasing when \( x < 0 \) and \( x > 4 \).
- \( f \) is decreasing when \( 0 < x < 4 \).
- \( f(x) > 0 \) when \(-2 < x < 3 \) and \( x > 5 \).
- \( f(x) < 0 \) when \( x < -2 \) and \( 3 < x < 5 \).

Use the graph to describe the degree and leading coefficient of \( f \).

**SOLUTION**

From the graph, \( f(x) \to -\infty \) as \( x \to -\infty \) and \( f(x) \to +\infty \) as \( x \to +\infty \).

So, the degree is odd and the leading coefficient is positive.

114 Chapter 3 Polynomial Functions
Solving a Real-Life Problem

The estimated number \( V \) (in thousands) of electric vehicles in use in the United States can be modeled by the polynomial function

\[
V(t) = 0.151280t^3 - 3.28234t^2 + 23.7565t - 2.041
\]

where \( t \) represents the year, with \( t = 1 \) corresponding to 2001.

a. Use a graphing calculator to graph the function for the interval \( 1 \leq t \leq 10 \).
Describe the behavior of the graph on this interval.

b. What was the average rate of change in the number of electric vehicles in use from 2001 to 2010?

c. Do you think this model can be used for years before 2001 or after 2010? Explain your reasoning.

SOLUTION

a. Using a graphing calculator and a viewing window of \( 1 \leq x \leq 10 \) and \( 0 \leq y \leq 65 \), you obtain the graph shown.

From 2001 to 2004, the numbers of electric vehicles in use increased. Around 2005, the growth in the numbers in use slowed and started to level off. Then the numbers in use started to increase again in 2009 and 2010.

b. The years 2001 and 2010 correspond to \( t = 1 \) and \( t = 10 \).

Average rate of change over \( 1 \leq t \leq 10 \):

\[
\frac{V(10) - V(1)}{10 - 1} = \frac{58.57 - 18.58444}{9} \approx 4.443
\]

The average rate of change from 2001 to 2010 is about 4.4 thousand electric vehicles per year.

c. Because the degree is odd and the leading coefficient is positive, \( V(t) \rightarrow -\infty \) as \( t \rightarrow -\infty \) and \( V(t) \rightarrow +\infty \) as \( t \rightarrow +\infty \). The end behavior indicates that the model has unlimited growth as \( t \) increases. While the model may be valid for a few years after 2010, in the long run, unlimited growth is not reasonable. Notice in 2000 that \( V(0) = -2.041 \). Because negative values of \( V(t) \) do not make sense given the context (electric vehicles in use), the model should not be used for years before 2001.

Monitoring Progress

Graph the polynomial function.

7. \( f(x) = x^4 + x^2 - 3 \)
8. \( f(x) = 4 - x^3 \)
9. \( f(x) = x^3 - x^2 + x - 1 \)

10. Sketch a graph of the polynomial function \( f \) having these characteristics.
    - \( f \) is decreasing when \( x < -1.5 \) and \( x > 2.5 \); \( f \) is increasing when \( -1.5 < x < 2.5 \).
    - \( f(x) > 0 \) when \( x < -3 \) and \( 1 < x < 4 \); \( f(x) < 0 \) when \( -3 < x < 1 \) and \( x > 4 \).
Use the graph to describe the degree and leading coefficient of \( f \).

11. WHAT IF? Repeat Example 6 using the alternative model for electric vehicles of

\[
V(t) = -0.0290900t^4 + 0.791260t^3 - 7.96583t^2 + 36.5561t - 12.025.
\]
3.1 Exercises  

Vocabulary and Core Concept Check

1. **WRITING** Explain what is meant by the end behavior of a polynomial function.

2. **WHICH ONE DOESN’T BELONG?** Which function does not belong with the other three? Explain your reasoning.

   - \( f(x) = 7x^5 + 3x^2 - 2x \)
   - \( g(x) = 3x^3 - 2x^8 + \frac{3}{4} \)
   - \( h(x) = -3x^4 + 5x^{-1} - 3x^2 \)
   - \( k(x) = \sqrt[3]{3x} + 8x^4 + 2x + 1 \)

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient. (See Example 1.)

3. \( f(x) = -3x + 5x^3 - 6x^2 + 2 \)
4. \( p(x) = \frac{1}{2}x^2 + 3x - 4x^3 + 6x^4 - 1 \)
5. \( f(x) = 9x^4 + 8x^3 - 6x^{-2} + 2x \)
6. \( g(x) = \sqrt[3]{3} - 12x + 13x^2 \)
7. \( h(x) = \frac{5}{3}x^3 - \sqrt{7}x^4 + 8x^3 - \frac{1}{2} + x \)
8. \( h(x) = 3x^4 + 2x - \frac{5}{x} + 9x^3 - 7 \)

ERROR ANALYSIS In Exercises 9 and 10, describe and correct the error in analyzing the function.

9. \( f(x) = 8x^3 - 7x^4 - 9x - 3x^2 + 11 \)
   - \( f \) is a polynomial function. 
The degree is 3 and \( f \) is a cubic function. 
The leading coefficient is 8.

10. \( f(x) = 2x^4 + 4x - 9\sqrt{x} + 3x^2 - 8 \)
   - \( f \) is a polynomial function. 
The degree is 4 and \( f \) is a quartic function. 
The leading coefficient is 2.

In Exercises 11–16, evaluate the function for the given value of \( x \). (See Example 2.)

11. \( h(x) = -3x^4 + 2x^3 - 12x - 6; x = -2 \)
12. \( f(x) = 7x^4 - 10x^2 + 14x - 26; x = -7 \)
13. \( g(x) = x^6 - 64x^4 + x^2 - 7x - 51; x = 8 \)
14. \( g(x) = -x^3 + 3x^2 + 5x + 1; x = -12 \)
15. \( p(x) = 2x^3 + 4x^2 + 6x + 7; x = \frac{1}{2} \)
16. \( h(x) = 5x^3 - 3x^2 + 2x + 4; x = -\frac{1}{3} \)

In Exercises 17–20, describe the end behavior of the graph of the function. (See Example 3.)

17. \( h(x) = -5x^4 + 7x^3 - 6x^2 + 9x + 2 \)
18. \( g(x) = 7x^7 + 12x^5 - 6x^3 - 2x - 18 \)
19. \( f(x) = -2x^4 + 12x^8 + 17 + 15x^3 \)
20. \( f(x) = 11 - 18x^2 - 5x^5 - 12x^4 - 2x \)

In Exercises 21 and 22, describe the degree and leading coefficient of the polynomial function using the graph.

21.

22.
23. USING STRUCTURE Determine whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.
\[ f(x) = 5x^3x + \frac{5}{2}x^3 - 9x^4 + \sqrt{2}x^2 + 4x - 1 - x^5x^4 - 4 \]

24. WRITING Let \( f(x) = 13 \). State the degree, type, and leading coefficient. Describe the end behavior of the function. Explain your reasoning.

In Exercises 25–32, graph the polynomial function. (See Example 4.)

25. \( p(x) = 3 - x^4 \)
26. \( g(x) = x^3 + x + 3 \)
27. \( f(x) = 4x - 9 - x^3 \)
28. \( p(x) = x^5 - 3x^3 + 2 \)
29. \( h(x) = x^3 - 2x^3 + 3x \)
30. \( h(x) = 5 + 3x^3 - x^4 \)
31. \( g(x) = x^6 - 3x^4 + 2x - 4 \)
32. \( p(x) = x^6 - 2x^5 - 2x^3 + x + 5 \)

ANALYZING RELATIONSHIPS In Exercises 33–36, describe the \( x \)-values for which (a) \( f \) is increasing or decreasing, (b) \( f(x) > 0 \), and (c) \( f(x) < 0 \).

33. 
34. 
35. 
36. 

In Exercises 37–40, sketch a graph of the polynomial function \( f \) having the given characteristics. Use the graph to describe the degree and leading coefficient of the function \( f \). (See Example 5.)

37. • \( f \) is increasing when \( x > 0.5 \); \( f \) is decreasing when \( x < 0.5 \).
   • \( f(x) > 0 \) when \( x < -2 \) and \( x > 3 \); \( f(x) < 0 \) when \( -2 < x < 3 \).

38. • \( f \) is increasing when \(-2 < x < 3 \); \( f \) is decreasing when \( x < -2 \) and \( x > 3 \).
   • \( f(x) > 0 \) when \( x < -4 \) and \( 1 < x < 5 \); \( f(x) < 0 \) when \(-4 < x < 1 \) and \( x > 5 \).

39. • \( f \) is increasing when \(-2 < x < 0 \) and \( x > 2 \); \( f \) is decreasing when \( x < -2 \) and \( 0 < x < 2 \).
   • \( f(x) > 0 \) when \( x < -3 \), \(-1 < x < 1 \), and \( x > 3 \);
   • \( f(x) < 0 \) when \(-3 < x < -1 \) and \( 1 < x < 3 \).

40. • \( f \) is increasing when \( x < -1 \) and \( x > 1 \); \( f \) is decreasing when \(-1 < x < 1 \).
   • \( f(x) > 0 \) when \(-1.5 < x < 0 \) and \( x > 1.5 \); \( f(x) < 0 \) when \( x < -1.5 \) and \( 0 < x < 1.5 \).

41. MODELING WITH MATHEMATICS From 1980 to 2007 the number of drive-in theaters in the United States can be modeled by the function
\[ d(t) = -0.141t^3 + 9.64t^2 - 232.5t + 2421 \]
where \( d(t) \) is the number of open theaters and \( t \) is the number of years after 1980. (See Example 6.)

a. Use a graphing calculator to graph the function for the interval \( 0 \leq t \leq 27 \). Describe the behavior of the graph on this interval.

b. What is the average rate of change in the number of drive-in movie theaters from 1980 to 1995 and from 1995 to 2007? Interpret the average rates of change.

c. Do you think this model can be used for years before 1980 or after 2007? Explain.

42. PROBLEM SOLVING The weight of an ideal round-cut diamond can be modeled by
\[ w = 0.00583d^3 - 0.0125d^2 + 0.022d - 0.01 \]
where \( w \) is the weight of the diamond (in carats) and \( d \) is the diamond (in millimeters). According to the model, what is the weight of a diamond with a diameter of 12 millimeters?
43. **ABSTRACT REASONING** Suppose \( f(x) \to \infty \) as \( x \to -\infty \) and \( f(x) \to -\infty \) as \( x \to \infty \). Describe the end behavior of \( g(x) = -f(x) \). Justify your answer.

44. **THOUGHT PROVOKING** Write an even degree polynomial function such that the end behavior of \( f \) is given by \( f(x) \to -\infty \) as \( x \to -\infty \) and \( f(x) \to -\infty \) as \( x \to \infty \). Justify your answer by drawing the graph of your function.

45. **USING TOOLS** In Section 1.2 Exercise 12, the function \( V = 4r^3(\pi + 4) \) represents the volume \( V \) of the tank. Use a graphing calculator to graph the function. Estimate the percent change in volume when \( r \) increases from 1 foot to 1 foot 1 inch. Is the percent change greater than you expected? Explain.

46. **MAKING AN ARGUMENT** Your friend uses the table to speculate that the function \( f \) is an even degree polynomial and the function \( g \) is an odd degree polynomial. Is your friend correct? Explain your reasoning.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-8</td>
<td>4113</td>
<td>497</td>
</tr>
<tr>
<td>-2</td>
<td>21</td>
<td>5</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>13</td>
<td>-3</td>
</tr>
<tr>
<td>8</td>
<td>4081</td>
<td>-495</td>
</tr>
</tbody>
</table>

47. **DRAWING CONCLUSIONS** The graph of a function is symmetric with respect to the y-axis if for each point \((a, b)\) on the graph, \((-a, b)\) is also a point on the graph. The graph of a function is symmetric with respect to the origin if for each point \((a, b)\) on the graph, \((-a, -b)\) is also a point on the graph.

**a.** Use a graphing calculator to graph the function \( y = x^n \) when \( n = 1, 2, 3, 4, 5, \) and 6. In each case, identify the symmetry of the graph.

**b.** Predict what symmetry the graphs of \( y = x^{10} \) and \( y = x^{11} \) each have. Explain your reasoning and then confirm your predictions by graphing.

48. **HOW DO YOU SEE IT?** The graph of a polynomial function is shown.

- **a.** Describe the degree and leading coefficient of \( f \).
- **b.** Describe the intervals where the function is increasing and decreasing.
- **c.** What is the constant term of the polynomial function?

49. **REASONING** A cubic polynomial function \( f \) has a leading coefficient of 2 and a constant term of \(-5\). When \( f(1) = 0 \) and \( f(2) = 3 \), what is \( f(-5) \)? Explain your reasoning.

50. **CRITICAL THINKING** The weight \( y \) (in pounds) of a rainbow trout can be modeled by \( y = 0.000304x^3 \), where \( x \) is the length (in inches) of the trout.

- **a.** Write a function that relates the weight \( y \) and length \( x \) of a rainbow trout when \( y \) is measured in kilograms and \( x \) is measured in centimeters. Use the fact that 1 kilogram \( \approx 2.20 \) pounds and 1 centimeter \( \approx 0.394 \) inch.

- **b.** Graph the original function and the function from part (a) in the same coordinate plane. What type of transformation can you apply to the graph of \( y = 0.000304x^3 \) to produce the graph from part (a)?

**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

<table>
<thead>
<tr>
<th>Simplify the expression. (Skills Review Handbook)</th>
</tr>
</thead>
<tbody>
<tr>
<td>51. ( xy + x^2 + 2xy + y^2 - 3x^2 )</td>
</tr>
<tr>
<td>52. ( 2h^3g + 3hg^3 + 7h^3g^2 + 5h^3g + 2hg^3 )</td>
</tr>
<tr>
<td>53. ( -wk + 3kz - 2kw + 9zk - kw )</td>
</tr>
<tr>
<td>54. ( a^2(m - 7a^3) - m(a^2 - 10) )</td>
</tr>
<tr>
<td>55. ( 3(xy - 4) + 3(4xy + 3) - xy(x^2y - 1) )</td>
</tr>
<tr>
<td>56. ( cv(9 - 3c) + 2c(v - 4c) + 6c )</td>
</tr>
</tbody>
</table>
3.2 Adding, Subtracting, and Multiplying Polynomials

**Essential Question**  How can you cube a binomial?

**EXPLORATION 1**  Cubing Binomials

Work with a partner. Find each product. Show your steps.

a. \((x + 1)^3 = (x + 1)(x + 1)^2\)  
   \[= (x + 1)\]  
   Multiply second power.  
   \[= \]  
   Multiply binomial and trinomial.  
   \[= \]  
   Write in standard form, \(ax^3 + bx^2 + cx + d\).

b. \((a + b)^3 = (a + b)(a + b)^2\)  
   \[= (a + b)\]  
   Multiply second power.  
   \[= \]  
   Multiply binomial and trinomial.  
   \[= \]  
   Write in standard form.

c. \((x - 1)^3 = (x - 1)(x - 1)^2\)  
   \[= (x - 1)\]  
   Multiply second power.  
   \[= \]  
   Multiply binomial and trinomial.  
   \[= \]  
   Write in standard form.

d. \((a - b)^3 = (a - b)(a - b)^2\)  
   \[= (a - b)\]  
   Multiply second power.  
   \[= \]  
   Multiply binomial and trinomial.  
   \[= \]  
   Write in standard form.

**LOOKING FOR STRUCTURE**

To be proficient in math, you need to look closely to discern a pattern or structure.

**EXPLORATION 2**  Generalizing Patterns for Cubing a Binomial

Work with a partner.

a. Use the results of Exploration 1 to describe a pattern for the coefficients of the terms when you expand the cube of a binomial. How is your pattern related to Pascal’s Triangle, shown at the right?

b. Use the results of Exploration 1 to describe a pattern for the exponents of the terms in the expansion of a cube of a binomial.

c. Explain how you can use the patterns you described in parts (a) and (b) to find the product \((2x - 3)^3\). Then find this product.

**Communicate Your Answer**

3. How can you cube a binomial?

4. Find each product.
   a. \((x + 2)^3\)  
   b. \((x - 2)^3\)  
   c. \((2x - 3)^3\)  
   d. \((x - 3)^3\)  
   e. \((-2x + 3)^3\)  
   f. \((3x - 5)^3\)
What You Will Learn
- Add and subtract polynomials.
- Multiply polynomials.
- Use Pascal’s Triangle and the Binomial Theorem to expand binomials.

Adding and Subtracting Polynomials
Recall that the set of integers is closed under addition and subtraction because every sum or difference results in an integer. To add or subtract polynomials, you add or subtract the coefficients of like terms. Because adding or subtracting polynomials results in a polynomial, the set of polynomials is closed under addition and subtraction.

EXAMPLE 1 Adding Polynomials Vertically and Horizontally

a. Add \(3x^3 + 2x^2 - x - 7\) and \(x^3 - 10x^2 + 8\) in a vertical format.
b. Add \(9y^3 + 3y^2 - 2y + 1\) and \(-5y^2 + y - 4\) in a horizontal format.

**SOLUTION**

a. \[3x^3 + 2x^2 - x - 7 + x^3 - 10x^2 + 8 = 4x^3 - 8x^2 - x + 1\]

b. \[(9y^3 + 3y^2 - 2y + 1) + (-5y^2 + y - 4) = 9y^3 + 3y^2 - 5y^2 - 2y + y + 1 - 4 = 9y^3 - 2y^2 - y - 3\]

To subtract one polynomial from another, add the opposite. To do this, change the sign of each term of the subtracted polynomial and then add the resulting like terms.

EXAMPLE 2 Subtracting Polynomials Vertically and Horizontally

a. Subtract \(2x^3 + 6x^2 - x + 1\) from \(8x^3 - 3x^2 - 2x + 9\) in a vertical format.
b. Subtract \(3z^2 + z - 4\) from \(2z^2 + 3z\) in a horizontal format.

**SOLUTION**

a. \[8x^3 - 3x^2 - 2x + 9 - (2x^3 + 6x^2 - x + 1) = 6x^3 - 9x^2 - x + 8\]

b. \[(2z^2 + 3z) - (3z^2 + z - 4) = -z^2 + 3z - 3z^2 - z + 4 = -2z^2 + 2z + 4\]

**Monitoring Progress**

Find the sum or difference.
1. \((2x^2 - 6x + 5) + (7x^2 - x - 9)\)
2. \((3t^3 + 8t^2 - t - 4) - (5t^3 - t^2 + 17)\)
Multiplying Polynomials

To multiply two polynomials, you multiply each term of the first polynomial by each term of the second polynomial. As with addition and subtraction, the set of polynomials is closed under multiplication.

**Example 3**  Multiplying Polynomials Vertically and Horizontally

a. Multiply \(-x^2 + 2x + 4\) and \(x - 3\) in a vertical format.

\[
\begin{array}{c}
  -x^2 + 2x + 4 \\
  \times \quad x - 3 \\
\hline
  3x^3 - 6x - 12 \\
  3x^2 + 2x + 4x \\
  -x^3 + 2x^2 - 2x - 12 \\
\end{array}
\]

Multiply \(-x^2 + 2x + 4\) by \(x - 3\).

Multiply \(-x^2 + 2x + 4\) by \(x\).

Combine like terms.

b. \((y + 5)(3y^2 - 2y + 2) = (y + 5)3y^2 - (y + 5)2y + (y + 5)2\)

\[
= 3y^3 + 15y^2 - 2y^2 - 10y + 2y + 10 \\
= 3y^3 + 13y^2 - 8y + 10
\]

**Example 4**  Multiplying Three Binomials

Multiply \(x - 1\), \(x + 4\), and \(x + 5\) in a horizontal format.

\[
(x - 1)(x + 4)(x + 5) = (x^2 + 3x - 4)(x + 5) \\
= (x^2 + 3x - 4)x + (x^2 + 3x - 4)5 \\
= x^3 + 3x^2 - 4x + 5x^2 + 15x - 20 \\
= x^3 + 8x^2 + 11x - 20
\]

Some binomial products occur so frequently that it is worth memorizing their patterns. You can verify these polynomial identities by multiplying.

---

**Core Concept**

Special Product Patterns

<table>
<thead>
<tr>
<th>Sum and Difference</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>((a + b)(a - b) = a^2 - b^2)</td>
<td>((x + 3)(x - 3) = x^2 - 9)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Square of a Binomial</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>((a + b)^2 = a^2 + 2ab + b^2)</td>
<td>((y + 4)^2 = y^2 + 8y + 16)</td>
</tr>
<tr>
<td>((a - b)^2 = a^2 - 2ab + b^2)</td>
<td>((2t - 5)^2 = 4t^2 - 20t + 25)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Cube of a Binomial</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>((a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3)</td>
<td>((z + 3)^3 = z^3 + 9z^2 + 27z + 27)</td>
</tr>
<tr>
<td>((a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3)</td>
<td>((m - 2)^3 = m^3 - 6m^2 + 12m - 8)</td>
</tr>
</tbody>
</table>
EXAMPLE 5 Proving a Polynomial Identity

a. Prove the polynomial identity for the cube of a binomial representing a sum:

\[(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.\]

b. Use the cube of a binomial in part (a) to calculate 113.

SOLUTION

a. Expand and simplify the expression on the left side of the equation.

\[(a + b)^3 = (a + b)(a + b)(a + b)\]
\[= (a^2 + 2ab + b^2)(a + b)\]
\[= (a^2 + 2ab + b^2)a + (a^2 + 2ab + b^2)b\]
\[= a^3 + a^2b + 2a^2b + 2ab^2 + ab^2 + b^3\]
\[= a^3 + 3a^2b + 3ab^2 + b^3 \checkmark\]

The simplified left side equals the right side of the original identity. So, the identity \((a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\) is true.

b. To calculate 113 using the cube of a binomial, note that 11 = 10 + 1.

\[11^3 = (10 + 1)^3\]
\[= 10^3 + 3(10)^2(1) + 3(10)(1)^2 + 1^3\]
\[= 1000 + 300 + 30 + 1\]
\[= 1331\]

EXAMPLE 6 Using Special Product Patterns

Find each product.

a. \((4n + 5)(4n - 5)\)  
b. \((9y - 2)^2\)  
c. \((ab + 4)^3\)

SOLUTION

a. \((4n + 5)(4n - 5) = (4n)^2 - 5^2\)
\[= 16n^2 - 25\]

b. \((9y - 2)^2 = (9y)^2 - 2(9y)(2) + 2^2\)
\[= 81y^2 - 36y + 4\]

c. \((ab + 4)^3 = (ab)^3 + 3(ab)^2(4) + 3(ab)(4)^2 + 4^3\)
\[= a^3b^3 + 12a^2b^2 + 48ab + 64\]

Monitoring Progress

Find the product.

3. \((4x^2 + x - 5)(2x + 1)\)  
4. \((y - 2)(5y^2 + 3y - 1)\)

5. \((m - 2)(m - 1)(m + 3)\)  
6. \((3r - 2)(3r + 2)\)

7. \((5a + 2)^2\)  
8. \((xy - 3)^3\)

9. (a) Prove the polynomial identity for the cube of a binomial representing a difference: \((a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3\).

(b) Use the cube of a binomial in part (a) to calculate 93.
Pascal’s Triangle and the Binomial Theorem

Consider the expansion of the binomial \((a + b)^n\) for whole number values of \(n\). When you arrange the coefficients of the variables in the expansion of \((a + b)^n\), you will see a special pattern called Pascal’s Triangle. Pascal’s Triangle is named after French mathematician Blaise Pascal (1623—1662).

### Pascal’s Triangle

In Pascal’s Triangle, the first and last numbers in each row are 1. Every number other than 1 is the sum of the closest two numbers in the row directly above it. The numbers in Pascal’s Triangle are the same numbers that are the coefficients of binomial expansions, as shown in the first six rows.

<table>
<thead>
<tr>
<th>(n)</th>
<th>((a + b)^n)</th>
<th>Binomial Expansion</th>
<th>(\text{Pascal’s Triangle})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0th row</td>
<td>0 ((a + b)^0 = 1)</td>
<td>(1)</td>
<td>(1)</td>
</tr>
<tr>
<td>1st row</td>
<td>1 ((a + b)^1 = a + b)</td>
<td>(1)</td>
<td>(1)</td>
</tr>
<tr>
<td>2nd row</td>
<td>2 ((a + b)^2 = a^2 + 2ab + b^2)</td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>3rd row</td>
<td>3 ((a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3)</td>
<td>(1)</td>
<td>(3)</td>
</tr>
<tr>
<td>4th row</td>
<td>4 ((a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)</td>
<td>(1)</td>
<td>(4)</td>
</tr>
<tr>
<td>5th row</td>
<td>5 ((a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5)</td>
<td>(1)</td>
<td>(5)</td>
</tr>
</tbody>
</table>

In general, the \(n\)th row in Pascal’s Triangle gives the coefficients of \((a + b)^n\). Here are some other observations about the expansion of \((a + b)^n\).

1. An expansion has \(n + 1\) terms.
2. The power of \(a\) begins with \(n\), decreases by 1 in each successive term, and ends with 0.
3. The power of \(b\) begins with 0, increases by 1 in each successive term, and ends with \(n\).
4. The sum of the powers of each term is \(n\).

### Example 7

**Using Pascal’s Triangle to Expand Binomials**

Use Pascal’s Triangle to expand (a) \((x - 2)^5\) and (b) \((3y + 1)^3\).

**SOLUTION**

**a.** The coefficients from the fifth row of Pascal’s Triangle are 1, 5, 10, 10, 5, and 1.

\[
(x - 2)^5 = 1x^5 + 5x^4(-2) + 10x^3(-2)^2 + 10x^2(-2)^3 + 5x(-2)^4 + 1(-2)^5
\]

\[
= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32
\]

**b.** The coefficients from the third row of Pascal’s Triangle are 1, 3, 3, and 1.

\[
(3y + 1)^3 = 1(3y)^3 + 3(3y)^2(1) + 3(3y)(1)^2 + 1(1)^3
\]

\[
= 27y^3 + 27y^2 + 9y + 1
\]

**Monitoring Progress**

10. Use Pascal’s Triangle to expand (a) \((z + 3)^4\) and (b) \((2t - 1)^5\).
The coefficients in the expansion of \((a + b)^n\) can also be represented using combinations.

\[
\begin{array}{ccc}
\text{n} & \text{Pascal's Triangle as Numbers} & \text{Pascal's Triangle as Combinations} & \text{Binomial Expansion} \\
0\text{th row} & 0 & C_0 & (a + b)^0 = 1 \\
1\text{st row} & 1 & 1 & C_1, C_2 & (a + b)^1 = a + b \\
2\text{nd row} & 1 & 2 & C_0, C_1, C_2 & (a + b)^2 = a^2 + 2ab + b^2 \\
3\text{rd row} & 3 & 3 & 3 & C_0, C_1, C_2, C_3 & (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3
\end{array}
\]

The results in the table are generalized in the **Binomial Theorem**.

### Core Concept

**The Binomial Theorem**

For any positive integer \(n\), the binomial expansion of \((a + b)^n\) is

\[(a + b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r,
\]

where \(r\) is an integer from 0 to \(n\).

**EXAMPLE 8** **Using the Binomial Theorem**

**a.** Use the Binomial Theorem to write the expansion of \((x^2 + y)^3\).

**b.** Find the coefficient of \(x^4\) in the expansion of \((3x + 2)^{10}\).

**SOLUTION**

**a.** \((x^2 + y)^3 = \binom{3}{0}x^6y^0 + \binom{3}{1}x^4y^1 + \binom{3}{2}x^2y^2 + \binom{3}{3}x^0y^3 = x^6 + 3x^4y + 3x^2y^2 + y^3\)

**b.** From the Binomial Theorem, you know

\[(3x + 2)^{10} = \sum_{r=0}^{10} \binom{10}{r}(3x)^{10-r}(2)^r.
\]

Each term in the expansion has the form \(\binom{10}{r}(3x)^{10-r}(2)^r\). The term containing \(x^4\) occurs when \(r = 6\).

\[\binom{10}{6}(3x)^4(2)^6 = (210)(81x^4)(64) = 1,088,640x^4\]

\(\text{The coefficient of } x^4 = 1,088,640\).

**Monitoring Progress**

11. Use the Binomial Theorem to write the expansion of \((a) (x + 3)^5\) and \((b) (2p - q)^3\).

12. Find the coefficient of \(x^5\) in the expansion of \((x - 3)^7\).

13. Find the coefficient of \(x^3\) in the expansion of \((2x + 5)^8\).
3.2 Exercises

Vocabulary and Core Concept Check

1. WRITING Describe three different methods to expand \((x + 3)^3\).
2. WRITING Is \((a + b)(a - b) = a^2 - b^2\) an identity? Explain your reasoning.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, find the sum. (See Example 1.)
3. \((3x^2 + 4x - 1) + (-2x^3 - 3x + 2)\)
4. \((-5x^2 + 4x - 2) + (-8x^2 + 2x + 1)\)
5. \((12x^3 - 3x^2 + 2x - 5) + (8x^4 - 3x^3 + 4x + 1)\)
6. \((8x^4 + 2x^2 - 1) + (3x^3 - 5x^2 + 7x + 1)\)
7. \((7x^6 + 2x^5 - 3x^2 + 9x) + (5x^5 + 8x^3 - 6x^2 + 2x - 5)\)
8. \((9x^4 - 3x^3 + 4x^2 + 5x + 7) + (11x^4 - 4x^3 - 11x - 9)\)

In Exercises 9–14, find the difference. (See Example 2.)
9. \((3x^3 - 2x^2 + 4x - 8) - (5x^3 + 12x^2 - 3x - 4)\)
10. \((7x^4 - 9x^3 - 4x^2 + 5x + 6) - (2x^4 + 3x^3 - x^2 + x - 4)\)
11. \((5x^6 - 2x^4 + 9x^3 + 2x - 4) - (7x^5 - 8x^4 + 2x - 11)\)
12. \((4x^5 - 7x^3 - 9x^2 + 18) - (14x^4 - 8x^4 + 11x^2 + x)\)
13. \((8x^6 + 6x^3 - 2x^2 + 10x) - (9x^5 - x^3 - 13x^2 + 4)\)
14. \((11x^4 - 9x^2 + 3x + 11) - (2x^4 + 6x^3 + 2x - 9)\)

15. MODELING WITH MATHEMATICS During a recent period of time, the numbers (in thousands) of males \(M\) and females \(F\) that attend degree-granting institutions in the United States can be modeled by
\[
M = 19.7t^2 + 310.5t + 7539.6 \\
F = 28t^2 + 368t + 10127.8
\]
where \(t\) is time in years.
Write a polynomial to model the total number of people attending degree-granting institutions. Interpret its constant term.

16. MODELING WITH MATHEMATICS A farmer plants a garden that contains corn and pumpkins. The total area (in square feet) of the garden is modeled by the expression \(2x^2 + 5x + 4\). The area of the corn is modeled by the expression \(x^2 - 3x + 2\). Write an expression that models the area of the pumpkins.

In Exercises 17–24, find the product. (See Example 3.)
17. \(7x^3(5x^2 + 3x + 1)\)
18. \((-4x^8)(11x^3 + 2x^2 + 9x + 1)\)
19. \((5x^2 - 4x + 6)(-2x + 3)\)
20. \((-x - 3)(2x^2 + 5x + 8)\)
21. \((x^3 - 2x - 4)(x^2 - 3x - 5)\)
22. \((3x^2 + x - 2)(-4x^2 - 2x - 1)\)
23. \((3x^3 - 9x + 7)(x^2 - 2x + 1)\)
24. \((4x^2 - 8x - 2)(x^4 + 3x^2 + 4x)\)

ERROR ANALYSIS In Exercises 25 and 26, describe and correct the error in performing the operation.

25. \((x^2 - 3x + 4) - (x^3 + 7x - 2)\)  
   \[x^2 - 3x + 4 - x^3 - 7x + 2\]  
   \[= -x^3 + x^2 + 6x + 6\]

26. \((2x - 7)^3 = (2x)^3 - 7^3\)  
   \[8x^3 - 343\]
In Exercises 27–32, find the product of the binomials. (See Example 4.)
27. \((x - 3)(x + 2)(x + 4)\)
28. \((x - 5)(x + 2)(x - 6)\)
29. \((x - 2)(3x + 1)(4x - 3)\)
30. \((2x + 5)(x - 2)(3x + 4)\)
31. \((3x - 4)(5 - 2x)(4x + 1)\)
32. \((4 - 5x)(1 - 2x)(3x + 2)\)
33. REASONING Prove the polynomial identity \((a + b)(a - b) = a^2 - b^2\). Then give an example of two whole numbers greater than 10 that can be multiplied using mental math and the given identity. Justify your answers. (See Example 5.)
34. NUMBER SENSE You have been asked to order textbooks for your class. You need to order 29 textbooks that cost $31 each. Explain how you can use the polynomial identity \((a + b)(a - b) = a^2 - b^2\) and mental math to find the total cost of the textbooks.

In Exercises 35–42, find the product. (See Example 6.)
35. \((x - 9)(x + 9)\)
36. \((m + 6)^2\)
37. \((3c - 5)^2\)
38. \((2y - 5)(2y + 5)\)
39. \((7h + 4)^2\)
40. \((9g - 4)^2\)
41. \((2k + 6)^3\)
42. \((4m - 3)^3\)

In Exercises 43–48, use Pascal’s Triangle to expand the binomial. (See Example 7.)
43. \((2r + 4)^3\)
44. \((6m + 2)^2\)
45. \((2q - 3)^4\)
46. \((g + 2)^5\)
47. \((yz + 1)^5\)
48. \((np - 1)^4\)

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In Exercises 49–52, find the \(n\)th row of Pascal’s Triangle. Then expand the binomial.
49. \(n = 6; (x + 3)^6\)
50. \(n = 7; (2x - t)^7\)
51. \(n = 9; (a + b^2)^9\)
52. \(n = 10; (y - 3z)^{10}\)
53. FINDING A PATTERN What is the sum of the numbers in each of rows 0–4 of Pascal’s Triangle? What is the sum of row \(n\)?

54. FINDING A PATTERN Describe the pattern formed by the sums of the numbers along the diagonal segments of Pascal’s Triangle. What is the name of this sequence of numbers?

In Exercises 55–62, use the Binomial Theorem to write the binomial expansion. (See Example 8a.)
55. \((x + 2)^3\)
56. \((c - 4)^5\)
57. \((a + 3b)^4\)
58. \((4p - q)^6\)
59. \((u^3 - 3)^4\)
60. \((2x^4 + 5)^5\)
61. \((3u + v^2)^6\)
62. \((x^3 - y^2)^4\)

In Exercises 63–70, use the given value of \(n\) to find the coefficient of \(x^n\) in the expansion of the binomial. (See Example 8b.)
63. \((x - 2)^{10}, n = 5\)
64. \((x - 3)^7, n = 4\)
65. \((x^2 - 3)^8, n = 6\)
66. \((3x + 2)^5, n = 3\)
67. \((2x + 5)^{12}, n = 7\)
68. \((3x - 1)^9, n = 2\)
69. \(\left(\frac{1}{2}x - 4\right)^{11}, n = 4\)
70. \(\left(\frac{1}{3}x + 6\right)^9, n = 3\)
71. **REASONING** Write the eighth row of Pascal’s Triangle as combinations and as numbers.

72. **PROBLEM SOLVING** The first four triangular numbers are 1, 3, 6, and 10.
   a. Use Pascal’s Triangle to write the first four triangular numbers as combinations.

   \[
   \begin{array}{c|c|c|c|c|c}
   1 & 1 & 1 & 1 & 1 & 1 \\
   1 & 2 & 1 & 1 & 1 & 1 \\
   1 & 3 & 3 & 1 & 1 & 1 \\
   1 & 4 & 6 & 4 & 1 & 1 \\
   1 & 5 & 10 & 10 & 5 & 1 \\
   \end{array}
   \]

   b. Use your result from part (a) to write an explicit rule for the \(n\)th triangular number \(T_n\).

**NUMBER SENSE** In Exercises 73–76, use the Binomial Theorem to approximate the quantity to the nearest thousandth. For example, in Exercise 73, use the expansion 

\[
(1.5)^6 = (1 + 0.5)^6 = 1 + 6(0.5) + 15(0.5)^2 \ldots
\]

73. \((1.6)^6\)

74. \((1.95)^7\)

75. \((2.99)^8\)

76. \((2.005)^9\)

**COMPARING METHODS** Find the product of the expression \((a^2 + 4b^2)(3a^2 - b^2)\) using two different methods. Which method do you prefer? Explain.

77. **COMPARING METHODS** Find the product of the expression \((a^2 + 4b^2)(3a^2 - b^2)\) using two different methods. Which method do you prefer? Explain.

78. **THOUGHT PROVOKING** Adjoin one or more polygons to the rectangle to form a single new polygon whose perimeter is double that of the rectangle. Find the perimeter of the new polygon.

\[
\text{Perimeter} = 2x + 3
\]

**MATHEMATICAL CONNECTIONS** In Exercises 79 and 80, write an expression for the volume of the figure as a polynomial in standard form.

79. \(V = lwh\)

80. \(V = \pi r^2 h\)

81. **MODELING WITH MATHEMATICS** Two people make three deposits into their bank accounts earning the same simple interest rate \(r\).

<table>
<thead>
<tr>
<th>Date</th>
<th>Transaction</th>
<th>Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>01/01/2012</td>
<td>Deposit</td>
<td>$2000.00</td>
</tr>
<tr>
<td>01/01/2013</td>
<td>Deposit</td>
<td>$3000.00</td>
</tr>
<tr>
<td>01/01/2014</td>
<td>Deposit</td>
<td>$1000.00</td>
</tr>
</tbody>
</table>

Person A

<table>
<thead>
<tr>
<th>Date</th>
<th>Transaction</th>
<th>Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>01/01/2012</td>
<td>Deposit</td>
<td>$5000.00</td>
</tr>
<tr>
<td>01/01/2013</td>
<td>Deposit</td>
<td>$1000.00</td>
</tr>
<tr>
<td>01/01/2014</td>
<td>Deposit</td>
<td>$4000.00</td>
</tr>
</tbody>
</table>

Person B

Person A’s account is worth

\[
2000(1 + r)^3 + 3000(1 + r)^2 + 1000(1 + r)
\]

on January 1, 2015.

a. Write a polynomial for the value of Person B’s account on January 1, 2015.

b. Write the total value of the two accounts as a polynomial in standard form. Then interpret the coefficients of the polynomial.

c. Suppose their interest rate is 0.05. What is the total value of the two accounts on January 1, 2015?

82. **USING STRUCTURE** Complete the table. What characteristic of Pascal’s Triangle does the table illustrate?

<table>
<thead>
<tr>
<th>(n)</th>
<th>(r)</th>
<th>(nC_n)</th>
<th>(nC_n - r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

83. **BINOMIAL EXPERIMENT** You roll a twelve-sided die 8 times. Let \(p\) be the probability of rolling a multiple of 3. Let \(q\) be the probability of rolling a number that is not a multiple of 3. The probability of rolling a multiple of 3 exactly \(k\) times in \(n\) rolls is

\[
nC_k p^k q^{n-k}.
\]

Expand \((\frac{2}{3} + \frac{1}{3})^8\) using the Binomial Theorem. Interpret each term in the context of the problem.
84. PROBLEM SOLVING
The sphere is centered in the cube. Find an expression for the volume of the cube outside the sphere.

85. MAKING AN ARGUMENT
Your friend claims the sum of two binomials is always a binomial and the product of two binomials is always a trinomial. Is your friend correct? Explain your reasoning.

86. HOW DO YOU SEE IT?
You make a tin box by cutting \( x \)-inch-by-\( x \)-inch pieces of tin off the corners of a rectangle and folding up each side. The plan for your box is shown.

```
  x   x   x   x
  x   x   6 - 2x
  x   12 - 2x
  x   x   x
```

a. What are the dimensions of the original piece of tin?
b. Write a function that represents the volume of the box. Without multiplying, determine its degree.

87. USING TOOLS
In Exercises 87–90, use a graphing calculator to make a conjecture about whether the two functions are equivalent. Explain your reasoning.

88. \( f(x) = (2x - 3)^3; \ g(x) = 8x^3 - 36x^2 + 54x - 27 \)

89. \( h(x) = (x + 2)^5; \ k(x) = x^5 + 10x^4 + 40x^3 + 80x^2 + 64x \)

90. \( f(x) = (-x - 3)^2; \ g(x) = x^2 + 12x^3 + 54x^2 + 108x + 80 \)

91. REWRITING EXPRESSIONS
Expand the complex number (a) \((1 + i)^2\) and (b) \((3 - i)^2\). Simplify your results using the fact that \(i = \sqrt{-1}\).

92. ABSTRACT REASONING
You are given the function \( f(x) = (x + a)(x + b)(x + c)(x + d) \). When \( f(x) \) is written in standard form, show that the coefficient of \( x^3 \) is the sum of \( a, b, c, \) and \( d \), and the constant term is the product of \( a, b, c, \) and \( d \).

93. DRAWING CONCLUSIONS
Let \( g(x) = 12x^3 + 8x + 9 \) and \( h(x) = 3x^3 + 2x^2 - 7x + 4 \).

a. What is the degree of the polynomial \( g(x) + h(x) \)?
b. What is the degree of the polynomial \( g(x) - h(x) \)?
c. What is the degree of the polynomial \( g(x) \cdot h(x) \)?
d. In general, if \( g(x) \) and \( h(x) \) are polynomials such that \( g(x) \) has degree \( m \) and \( h(x) \) has degree \( n \), and \( m > n \), what are the degrees of \( g(x) + h(x) \), \( g(x) - h(x) \), and \( g(x) \cdot h(x) \)?

94. FINDING A PATTERN
In this exercise, you will explore the sequence of square numbers. The first four square numbers are represented below.

```
1   4   9   16
```

a. Find the differences between consecutive square numbers. Explain what you notice.
b. Show how the polynomial identity \((n + 1)^2 - n^2 = 2n + 1\) models the differences between square numbers.
c. Prove the polynomial identity in part (b).

95. CRITICAL THINKING
Recall that a Pythagorean triple is a set of positive integers \( a, b, \) and \( c \) such that \( a^2 + b^2 = c^2 \). The numbers 3, 4, and 5 form a Pythagorean triple because \( 3^2 + 4^2 = 5^2 \). You can use the polynomial identity \((x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2\) to generate other Pythagorean triples.

a. Prove the polynomial identity is true by showing that the simplified expressions for the left and right sides are the same.
b. Use the identity to generate the Pythagorean triple when \( x = 6 \) and \( y = 5 \).
c. Verify that your answer in part (b) satisfies \( a^2 + b^2 = c^2 \).

**Maintaining Mathematical Proficiency**
Reviewing what you learned in previous grades and lessons

96. \((3 - 2i) + (5 + 9i)\)
97. \((12 + 3i) - (7 - 8i)\)
98. \((7i)(-3i)\)
99. \((4+i)(2-i)\)

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3.3 Dividing Polynomials

Essential Question  How can you use the factors of a cubic polynomial to solve a division problem involving the polynomial?

Exploration 1  Dividing Polynomials

Work with a partner. Match each division statement with the graph of the related cubic polynomial $f(x)$. Explain your reasoning. Use a graphing calculator to verify your answers.

a. $\frac{f(x)}{x} = (x - 1)(x + 2)$

b. $\frac{f(x)}{x - 1} = (x - 1)(x + 2)$

c. $\frac{f(x)}{x + 1} = (x - 1)(x + 2)$

d. $\frac{f(x)}{x - 2} = (x - 1)(x + 2)$

e. $\frac{f(x)}{x + 2} = (x - 1)(x + 2)$

f. $\frac{f(x)}{x - 3} = (x - 1)(x + 2)$

A.  

B.  

C.  

D.  

E.  

F.  

Reasoning Abstractly

To be proficient in math, you need to understand a situation abstractly and represent it symbolically.

Exploration 2  Dividing Polynomials

Work with a partner. Use the results of Exploration 1 to find each quotient. Write your answers in standard form. Check your answers by multiplying.

a. $(x^3 + x^2 - 2x) + x$

b. $(x^3 - 3x + 2) + (x - 1)$

c. $(x^3 + 2x^2 - x - 2) + (x + 1)$

d. $(x^3 - x^2 - 4x + 4) + (x - 2)$

e. $(x^3 + 3x^2 - 4) + (x + 2)$

f. $(x^3 - 2x^2 - 5x + 6) + (x - 3)$

Communicate Your Answer

3. How can you use the factors of a cubic polynomial to solve a division problem involving the polynomial?
3.3 Lesson

What You Will Learn

- Use long division to divide polynomials by other polynomials.
- Use synthetic division to divide polynomials by binomials of the form \( x - k \).
- Use the Remainder Theorem.

Long Division of Polynomials

When you divide a polynomial \( f(x) \) by a nonzero polynomial divisor \( d(x) \), you get a quotient polynomial \( q(x) \) and a remainder polynomial \( r(x) \).

\[
\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}
\]

The degree of the remainder must be less than the degree of the divisor. When the remainder is 0, the divisor divides evenly into the dividend. Also, the degree of the divisor is less than or equal to the degree of the dividend \( f(x) \). One way to divide polynomials is called polynomial long division.

**Example 1** Using Polynomial Long Division

Divide \( 2x^4 + 3x^3 + 5x - 1 \) by \( x^2 + 3x + 2 \).

**SOLUTION**

Write polynomial division in the same format you use when dividing numbers. Include a “0” as the coefficient of \( x^2 \) in the dividend. At each stage, divide the term with the highest power in what is left of the dividend by the first term of the divisor. This gives the next term of the quotient.

\[
\begin{array}{c|ccccc}
& 2x^2 & - & 3x & + & 5 \\
\hline
x^2 + 3x + 2 & 2x^4 & + & 3x^3 & + & 0x^2 & + & 5x & - & 1 \\
& 2x^4 & + & 6x^3 & + & 4x^2 & & & & \\
\hline
& & -3x^3 & - & 4x^2 & + & 5x & & & \\
& & -3x^3 & - & 9x^2 & - & 6x & & & \\
\hline
& & & & 5x^2 & + & 11x & - & 1 & \\
& & & & 5x^2 & + & 15x & + & 10 & \\
\hline
& & & & & & -4x & - & 11 & &
\end{array}
\]

The expression added to the quotient in the result of a long division problem is \( \frac{r(x)}{d(x)} \) not \( r(x) \).

**Check**

You can check the result of a division problem by multiplying the quotient by the divisor and adding the remainder. The result should be the dividend.

\[
(2x^2 - 3x + 5)(x^2 + 3x + 2) + (-4x - 11)
\]

\[
= (2x^2)(x^2 + 3x + 2) - (3x)(x^2 + 3x + 2) + (5)(x^2 + 3x + 2) - 4x - 11
\]

\[
= 2x^4 + 6x^3 + 4x^2 - 3x^3 - 9x^2 - 6x + 5x^2 + 15x + 10 - 4x - 11
\]

\[
= 2x^4 + 3x^3 + 5x - 1
\]

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**Monitoring Progress**

Divide using polynomial long division.

1. \( (x^3 - x^2 - 2x + 8) \div (x - 1) \)
2. \( (x^4 + 2x^2 - x + 5) \div (x^2 - x + 1) \)
Synthetic Division

There is a shortcut for dividing polynomials by binomials of the form \(x - k\). This shortcut is called synthetic division. This method is shown in the next example.

**EXAMPLE 2** Using Synthetic Division

Divide \(-x^3 + 4x^2 + 9\) by \(x - 3\).

**SOLUTION**

**Step 1** Write the coefficients of the dividend in order of descending exponents. Include a “0” for the missing \(x\)-term. Because the divisor is \(x - 3\), use \(k = 3\). Write the \(k\)-value to the left of the vertical bar.

\[
\begin{array}{c|cccc}
\text{k-value} & 3 & -1 & 4 & 0 & 9 \\
\hline
\end{array}
\]

**Step 2** Bring down the leading coefficient. Multiply the leading coefficient by the \(k\)-value. Write the product under the second coefficient. Add.

\[
\begin{array}{c|cccc}
\text{coefficients of quotient} & 3 & -1 & 4 & 0 & 9 \\
\hline
-3 & -1 & 1 & 3 & 9 \\
\hline
1 & 1 & 3 & 18 \\
\end{array}
\]

**Step 3** Multiply the previous sum by the \(k\)-value. Write the product under the third coefficient. Add. Repeat this process for the remaining coefficient. The first three numbers in the bottom row are the coefficients of the quotient, and the last number is the remainder.

\[
\begin{array}{c|cccc}
\text{coefficients of quotient} & 3 & -1 & 4 & 0 & 9 \\
\hline
-3 & -1 & 1 & 3 & 9 \\
\hline
1 & 1 & 3 & 18 \\
\hline
\end{array}
\]

\[
\frac{-x^3 + 4x^2 + 9}{x - 3} = -x^2 + x + 3 + \frac{18}{x - 3}
\]

**EXAMPLE 3** Using Synthetic Division

Divide \(3x^3 - 2x^2 + 2x - 5\) by \(x + 1\).

**SOLUTION**

Use synthetic division. Because the divisor is \(x + 1 = x - (-1)\), \(k = -1\).

\[
\begin{array}{c|cccc}
-1 & 3 & -2 & 2 & -5 \\
\hline
3 & -5 & 7 & -12 \\
\hline
3 & -2 & 2 & -5 \\
\end{array}
\]

\[
\frac{3x^3 - 2x^2 + 2x - 5}{x + 1} = 3x^2 - 5x + 7 - \frac{12}{x + 1}
\]

**Monitoring Progress** Help in English and Spanish at BigIdeasMath.com

Divide using synthetic division.

3. \((x^3 - 3x^2 - 7x + 6) \div (x - 2)\)
4. \((2x^3 - x - 7) \div (x + 3)\)
The Remainder Theorem

The remainder in the synthetic division process has an important interpretation. When you divide a polynomial \( f(x) \) by \( d(x) = x - k \), the result is

\[
\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}
\]

Polynomial division

\[
\frac{f(x)}{x - k} = q(x) + \frac{r(x)}{x - k}
\]

Substitute \( x - k \) for \( d(x) \).

\[
f(x) = (x - k)q(x) + r(x).
\]

Multiply both sides by \( x - k \). Because either \( r(x) = 0 \) or the degree of \( r(x) \) is less than the degree of \( x - k \), you know that \( r(x) \) is a constant function. So, let \( r(x) = r \), where \( r \) is a real number, and evaluate \( f(x) \) when \( x = k \).

\[
f(k) = (k - k)q(k) + r
\]

Substitute \( k \) for \( x \) and \( r \) for \( r(x) \).

\[
f(k) = r
\]

Simplify. This result is stated in the Remainder Theorem.

Core Concept

The Remainder Theorem

If a polynomial \( f(x) \) is divided by \( x - k \), then the remainder is \( r = f(k) \).

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. So, to evaluate \( f(x) \) when \( x = k \), divide \( f(x) \) by \( x - k \). The remainder will be \( f(k) \).

**EXAMPLE 4** Evaluating a Polynomial

Use synthetic division to evaluate \( f(x) = 5x^3 - x^2 + 13x + 29 \) when \( x = -4 \).

**SOLUTION**

\[
\begin{array}{c|cccc}
-4 & 5 & -1 & 13 & 29 \\
 & & -20 & 84 & -388 \\
\hline
 & 5 & -21 & 97 & -359 \\
\end{array}
\]

The remainder is \(-359\). So, you can conclude from the Remainder Theorem that \( f(-4) = -359 \).

**Check**

Check this by substituting \( x = -4 \) in the original function.

\[
f(-4) = 5(-4)^3 - (-4)^2 + 13(-4) + 29
\]

\[
= -320 - 16 - 52 + 29
\]

\[
= -359 \quad \checkmark
\]

**Monitoring Progress** Help in English and Spanish at BigIdeasMath.com

Use synthetic division to evaluate the function for the indicated value of \( x \).

5. \( f(x) = 4x^2 - 10x - 21; x = 5 \)

6. \( f(x) = 5x^4 + 2x^3 - 20x - 6; x = 2 \)
3.3 Exercises
Dynamic Solutions available at BigIdeasMath.com

Vocabulary and Core Concept Check

1. WRITING Explain the Remainder Theorem in your own words. Use an example in your explanation.

2. VOCABULARY What form must the divisor have to make synthetic division an appropriate method for dividing a polynomial? Provide examples to support your claim.

3. VOCABULARY Write the polynomial divisor, dividend, and quotient functions represented by the synthetic division shown at the right.

4. WRITING Explain what the colored numbers represent in the synthetic division in Exercise 3.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–10, divide using polynomial long division. (See Example 1.)

5. \((x^2 + x - 17) ÷ (x - 4)\)

6. \((3x^2 - 14x - 5) ÷ (x - 5)\)

7. \((x^3 + x^2 + x + 2) ÷ (x^2 - 1)\)

8. \((7x^3 + x^2 + x) ÷ (x^2 + 1)\)

9. \((5x^4 - 2x^3 - 7x^2 - 39) ÷ (x^2 + 2x - 4)\)

10. \((4x^4 + 5x - 4) ÷ (x^2 - 3x - 2)\)

In Exercises 11–18, divide using synthetic division. (See Examples 2 and 3.)

11. \((x^2 + 8x + 1) ÷ (x - 4)\)

12. \((4x^2 - 13x - 5) ÷ (x - 2)\)

13. \((2x^2 - x + 7) ÷ (x + 5)\)

14. \((x^3 - 4x + 6) ÷ (x + 3)\)

15. \((x^2 + 9) ÷ (x - 3)\)

16. \((3x^3 - 5x^2 - 2) ÷ (x - 1)\)

17. \((x^4 - 5x^3 - 8x^2 + 13x - 12) ÷ (x - 6)\)

18. \((x^4 + 4x^3 + 16x - 35) ÷ (x + 5)\)

ANALYZING RELATIONSHIPS In Exercises 19–22, match the equivalent expressions. Justify your answers.

19. \((x^2 + x - 3) ÷ (x - 2)\)

20. \((x^2 - x - 3) ÷ (x - 2)\)

21. \((x^2 - x + 3) ÷ (x - 2)\)

22. \((x^2 + x + 3) ÷ (x - 2)\)

A. \(x + 1 - \frac{1}{x - 2}\)

B. \(x + 3 + \frac{9}{x - 2}\)

C. \(x + 1 + \frac{5}{x - 2}\)

D. \(x + 3 + \frac{3}{x - 2}\)

ERROR ANALYSIS In Exercises 23 and 24, describe and correct the error in using synthetic division to divide \(x^3 - 5x + 3\) by \(x - 2\).

23.

24.
In Exercises 25–32, use synthetic division to evaluate the function for the indicated value of $x$. (See Example 4.)

- 25. $f(x) = -x^2 - 8x + 30; x = -1$
- 26. $f(x) = 3x^2 + 2x - 20; x = 3$
- 27. $f(x) = x^3 - 2x^2 + 4x + 3; x = 2$
- 28. $f(x) = x^3 + x^2 - 3x + 9; x = -4$
- 29. $f(x) = x^3 - 6x + 1; x = 6$
- 30. $f(x) = x^3 - 9x - 7; x = 10$
- 31. $f(x) = x^4 + 6x^2 - 7x + 1; x = 3$
- 32. $f(x) = -x^4 - x^3 - 2; x = 5$

33. **MAKING AN ARGUMENT** You use synthetic division to divide $f(x)$ by $(x - a)$ and find that the remainder equals 15. Your friend concludes that $f(15) = a$. Is your friend correct? Explain your reasoning.

34. **THOUGHT PROVOKING** A polygon has an area represented by $A = 4x^2 + 8x + 4$. The figure has at least one dimension equal to $2x + 2$. Draw the figure and label its dimensions.

35. **USING TOOLS** The total attendance $A$ (in thousands) at NCAA women’s basketball games and the number $T$ of NCAA women’s basketball teams over a period of time can be modeled by

\[
A = -1.95x^3 + 70.1x^2 - 188x + 2150
\]

\[
T = 14.8x + 725
\]

where $x$ is in years and $0 < x < 18$. Write a function for the average attendance per team over this period of time.

36. **COMPARING METHODS** The profit $P$ (in millions of dollars) for a DVD manufacturer can be modeled by

\[
P = -6x^3 + 72x,
\]

where $x$ is the number (in millions) of DVDs produced. Use synthetic division to show that the company yields a profit of $96 million when 2 million DVDs are produced. Is there an easier method? Explain.

37. **CRITICAL THINKING** What is the value of $k$ such that

\[
(x^3 - x^2 + kx - 30) ÷ (x - 5)
\]

has a remainder of zero? 

A) $-14$  B) $-2$  C) $26$  D) $32$

38. **HOW DO YOU SEE IT?** The graph represents the polynomial function $f(x) = x^3 + 3x^2 - x - 3$.

![Graph](image)

- a. The expression $f(x) ÷ (x - k)$ has a remainder of $-15$. What is the value of $k$?
- b. Use the graph to compare the remainders of

\[
(x^3 + 3x^2 - x - 3) ÷ (x + 3)
\]

and

\[
(x^3 + 3x^2 - x - 3) ÷ (x + 1)
\]

39. **MATHEMATICAL CONNECTIONS** The volume $V$ of the rectangular prism is given by

\[
V = 2x^3 + 17x^2 + 46x + 40
\]

Find an expression for the missing dimension.

![Diagram](image)

40. **USING STRUCTURE** You divide two polynomials and obtain the result $5x^2 - 13x + 47 - \frac{102}{x + 2}$. What is the dividend? How did you find it?

---

**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

Find the zero(s) of the function. *(Skills Review Handbook)*

- 41. $f(x) = x^2 - 6x + 9$
- 42. $g(x) = 3(x + 6)(x - 2)$
- 43. $g(x) = x^2 + 14x + 49$
- 44. $h(x) = 4x^2 + 36$

---

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3.4 Factoring Polynomials

**Essential Question**  How can you factor a polynomial?

**Exploration 1**

**Factoring Polynomials**

**Work with a partner.** Match each polynomial equation with the graph of its related polynomial function. Use the x-intercepts of the graph to write each polynomial in factored form. Explain your reasoning.

- a. \(x^2 + 5x + 4 = 0\)
- b. \(x^3 - 2x^2 - x + 2 = 0\)
- c. \(x^3 + x^2 - 2x = 0\)
- d. \(x^3 - x = 0\)
- e. \(x^4 - 5x^2 + 4 = 0\)
- f. \(x^4 - 2x^3 - x^2 + 2x = 0\)

**Exploration 2**

**Factoring Polynomials**

**Work with a partner.** Use the x-intercepts of the graph of the polynomial function to write each polynomial in factored form. Explain your reasoning. Check your answers by multiplying.

- a. \(f(x) = x^2 - x - 2\)
- b. \(f(x) = x^3 - x^2 - 2x\)
- c. \(f(x) = x^3 - 2x^2 - 3x\)
- d. \(f(x) = x^3 - 3x^2 - x + 3\)
- e. \(f(x) = x^4 + 2x^3 - x^2 - 2x\)
- f. \(f(x) = x^4 - 10x^2 + 9\)

**Communicate Your Answer**

3. How can you factor a polynomial?

4. What information can you obtain about the graph of a polynomial function written in factored form?
**What You Will Learn**

- Factor polynomials.
- Use the Factor Theorem.

**Factoring Polynomials**

Previously, you factored quadratic polynomials. You can also factor polynomials with degree greater than 2. Some of these polynomials can be factored completely using techniques you have previously learned. A factorable polynomial with integer coefficients is factored completely when it is written as a product of unfactorable polynomials with integer coefficients.

### Example 1
**Finding a Common Monomial Factor**

Factor each polynomial completely.

a. \(x^3 - 4x^2 - 5x\)  
   \(3y^5 - 48y^3\)  
   \(5z^4 + 30z^3 + 45z^2\)

**SOLUTION**

a. \(x^3 - 4x^2 - 5x = x(x^2 - 4x - 5)\)  
   Factor common monomial.  
   \(= x(x - 5)(x + 1)\)  
   Factor trinomial.

b. \(3y^5 - 48y^3 = 3y^3(y^2 - 16)\)  
   Factor common monomial.  
   \(= 3y^3(y - 4)(y + 4)\)  
   Difference of Two Squares Pattern

c. \(5z^4 + 30z^3 + 45z^2 = 5z^2(z^2 + 6z + 9)\)  
   Factor common monomial.  
   \(= 5z^2(z + 3)^2\)  
   Perfect Square Trinomial Pattern

**Monitoring Progress**

Factor the polynomial completely.

1. \(x^3 - 7x^2 + 10x\)  
   2. \(3n^3 - 75n^n\)  
   3. \(8m^5 - 16m^4 + 8m^3\)

In part (b) of Example 1, the special factoring pattern for the difference of two squares was used to factor the expression completely. There are also factoring patterns that you can use to factor the sum or difference of two cubes.

**Core Concept**

**Special Factoring Patterns**

**Sum of Two Cubes**

\[a^3 + b^3 = (a + b)(a^2 - ab + b^2)\]

**Example**

\[64x^3 + 1 = (4x)^3 + 1^3\]

\[= (4x + 1)(16x^2 - 4x + 1)\]

**Difference of Two Cubes**

\[a^3 - b^3 = (a - b)(a^2 + ab + b^2)\]

**Example**

\[27x^3 - 8 = (3x)^3 - 2^3\]

\[= (3x - 2)(9x^2 + 6x + 4)\]
**EXAMPLE 2**  Factoring the Sum or Difference of Two Cubes

Factor (a) \(x^3 - 125\) and (b) \(16s^5 + 54s^2\) completely.

**SOLUTION**

a. \(x^3 - 125 = x^3 - 5^3\)

\[= (x - 5)(x^2 + 5x + 25)\]

Write as \(a^3 - b^3\). Difference of Two Cubes Pattern

b. \(16s^5 + 54s^2 = 2s^2(8s^3 + 27)\)

\[= 2s^2(2s + 3)(4s^2 - 6s + 9)\]

Factor common monomial. Write \(8s^3 + 27\) as \(a^3 + b^3\). Sum of Two Cubes Pattern

For some polynomials, you can factor by grouping pairs of terms that have a common monomial factor. The pattern for factoring by grouping is shown below.

\[ra + rb + sa + sb = r(a + b) + s(a + b)\]

\[= (r + s)(a + b)\]

**EXAMPLE 3**  Factoring by Grouping

Factor \(z^3 + 5z^2 - 4z - 20\) completely.

**SOLUTION**

\[z^3 + 5z^2 - 4z - 20 = z^3(z + 5) - 4(z + 5)\]

Factor by grouping. Distributive Property

\[= (z^3 - 4)(z + 5)\]

Difference of Two Squares Pattern

An expression of the form \(au^2 + bu + c\), where \(u\) is an algebraic expression, is said to be in quadratic form. The factoring techniques you have studied can sometimes be used to factor such expressions.

**EXAMPLE 4**  Factoring Polynomials in Quadratic Form

Factor (a) \(16x^4 - 81\) and (b) \(3p^8 + 15p^5 + 18p^2\) completely.

**SOLUTION**

a. \(16x^4 - 81 = (4x^2)^2 - 9^2\)

\[= (4x^2 + 9)(4x^2 - 9)\]

Write as \(a^2 - b^2\). Difference of Two Squares Pattern

\[= (4x^2 + 9)(2x - 3)(2x + 3)\]

Difference of Two Squares Pattern

b. \(3p^8 + 15p^5 + 18p^2 = 3p^5(p^3 + 5p^2 + 6)\)

Factor common monomial. Factor trinomial in quadratic form.

\[= 3p^5(p^3 + 3)(p^3 + 2)\]

**Monitoring Progress**

Factor the polynomial completely.

4. \(a^3 + 27\)  
5. \(6z^5 - 750z^2\)
6. \(x^3 + 4x^2 - x - 4\)  
7. \(3y^3 + y^2 + 9y + 3\)
8. \(-16x^4 + 625\)  
9. \(5w^6 - 25w^4 + 30w^2\)

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The Factor Theorem
When dividing polynomials in the previous section, the examples had nonzero remainders. Suppose the remainder is 0 when a polynomial \( f(x) \) is divided by \( x - k \). Then,

\[
\frac{f(x)}{x - k} = q(x) + \frac{0}{x - k} = q(x)
\]

where \( q(x) \) is the quotient polynomial. Therefore, \( f(x) = (x - k) \cdot q(x) \), so that \( x - k \) is a factor of \( f(x) \). This result is summarized by the Factor Theorem, which is a special case of the Remainder Theorem.

**Core Concept**

**The Factor Theorem**
A polynomial \( f(x) \) has a factor \( x - k \) if and only if \( f(k) = 0 \).

**Example 5**

**Determining Whether a Linear Binomial Is a Factor**

Determine whether (a) \( x - 2 \) is a factor of \( f(x) = x^2 + 2x - 4 \) and (b) \( x + 5 \) is a factor of \( f(x) = 3x^4 + 15x^3 - x^2 + 25 \).

**Solution**

\textbf{a.} Find \( f(2) \) by direct substitution.

\[
f(2) = 2^2 + 2(2) - 4 = 4 + 4 - 4 = 4
\]

Because \( f(2) \neq 0 \), the binomial \( x - 2 \) is not a factor of \( f(x) = x^2 + 2x - 4 \).

\textbf{b.} Find \( f(-5) \) by synthetic division.

\[
\begin{array}{c|cccc}
-5 & 3 & 15 & -1 & 0 \\
& 3 & 0 & -1 & 5 \\
\hline
3 & 0 & -1 & 5 & 0
\end{array}
\]

Because \( f(-5) = 0 \), the binomial \( x + 5 \) is a factor of \( f(x) = 3x^4 + 15x^3 - x^2 + 25 \).

**Example 6**

**Factoring a Polynomial**

Show that \( x + 3 \) is a factor of \( f(x) = x^4 + 3x^3 - x - 3 \). Then factor \( f(x) \) completely.

**Solution**

Show that \( f(-3) = 0 \) by synthetic division.

\[
\begin{array}{c|cccc}
-3 & 1 & 3 & 0 & -1 & 3 \\
& -3 & 0 & 0 & 3 \\
\hline
1 & 0 & 0 & -1 & 0
\end{array}
\]

Because \( f(-3) = 0 \), you can conclude that \( x + 3 \) is a factor of \( f(x) \) by the Factor Theorem. Use the result to write \( f(x) \) as a product of two factors and then factor completely.

\[
f(x) = x^4 + 3x^3 - x - 3 = (x + 3)(x^3 - 1) = (x + 3)(x - 1)(x^2 + x + 1)
\]

Difference of Two Cubes Pattern
Because the x-intercepts of the graph of a function are the zeros of the function, you can use the graph to approximate the zeros. You can check the approximations using the Factor Theorem.

**EXAMPLE 7**  Real-Life Application

During the first 5 seconds of a roller coaster ride, the function \( h(t) = 4t^3 - 21t^2 + 9t + 34 \) represents the height \( h \) (in feet) of the roller coaster after \( t \) seconds. How long is the roller coaster at or below ground level in the first 5 seconds?

**SOLUTION**

1. **Understand the Problem**  You are given a function rule that represents the height of a roller coaster. You are asked to determine how long the roller coaster is at or below ground during the first 5 seconds of the ride.

2. **Make a Plan**  Use a graph to estimate the zeros of the function and check using the Factor Theorem. Then use the zeros to describe where the graph lies below the \( t \)-axis.

3. **Solve the Problem**  From the graph, two of the zeros appear to be \(-1\) and \(2\). The third zero is between 4 and 5.

   **Step 1**  Determine whether \(-1\) is a zero using synthetic division.

   \[
   \begin{array}{c|cccc}
   -1 & 4 & -21 & 9 & 34 \\
   0 & -4 & 25 & -34 \\
   \hline
   & 4 & -25 & 34 & 0
   \end{array}
   \]

   \( h(-1) = 0 \), so \(-1\) is a zero of \( h \) and \( t + 1 \) is a factor of \( h(t) \).

   **Step 2**  Determine whether 2 is a zero. If 2 is also a zero, then \( t - 2 \) is a factor of the resulting quotient polynomial. Check using synthetic division.

   \[
   \begin{array}{c|cccc}
   2 & 4 & -25 & 34 \\
   0 & 8 & -34 \\
   \hline
   & 4 & -17 & 0
   \end{array}
   \]

   The remainder is 0, so \( t - 2 \) is a factor of \( h(t) \) and 2 is a zero of \( h \).

   So, \( h(t) = (t + 1)(t - 2)(4t - 17) \). The factor \( 4t - 17 \) indicates that the zero between 4 and 5 is \( \frac{17}{4} \), or 4.25.

   The zeros are \(-1\), 2, and 4.25. Only \( t = 2 \) and \( t = 4.25 \) occur in the first 5 seconds. The graph shows that the roller coaster is at or below ground level for \( 4.25 \) \(- 2 = 2.25 \) seconds.

4. **Look Back**  Use a table of values to verify the positive zeros and heights between the zeros.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( h(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.25</td>
<td>20.25</td>
</tr>
<tr>
<td>2.75</td>
<td>20.25</td>
</tr>
<tr>
<td>3.5</td>
<td>-16.88</td>
</tr>
<tr>
<td>4.25</td>
<td>-20.25</td>
</tr>
<tr>
<td>5</td>
<td>54</td>
</tr>
</tbody>
</table>

**Monitoring Progress**

10. Determine whether \( x - 4 \) is a factor of \( f(x) = 2x^2 + 5x - 12 \).

11. Show that \( x - 6 \) is a factor of \( f(x) = x^3 - 5x^2 - 6x \). Then factor \( f(x) \) completely.

12. In Example 7, does your answer change when you first determine whether 2 is a zero and then whether \(-1\) is a zero? Justify your answer.

---

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3.4  Exercises

Vocabulary and Core Concept Check

1. COMPLETE THE SENTENCE The expression \(9x^4 - 49\) is in _________ form because it can be written as \(u^2 - 49\) where \(u = \) ________.

2. VOCABULARY Explain when you should try factoring a polynomial by grouping.

3. WRITING How do you know when a polynomial is factored completely?

4. WRITING Explain the Factor Theorem and why it is useful.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, factor the polynomial completely. (See Example 1.)

5. \(x^3 - 2x^2 - 24x\)
6. \(4k^5 - 100k^3\)
7. \(3p^5 - 192p^3\)
8. \(2m^6 - 24m^5 + 64m^4\)
9. \(2q^4 + 9q^3 - 18q^2\)
10. \(3r^6 - 11r^5 - 20r^4\)
11. \(10w^{10} - 19w^9 + 6w^8\)
12. \(18t^9 + 33t^8 + 14t^7\)

In Exercises 13–20, factor the polynomial completely. (See Example 2.)

13. \(x^3 + 64\)
14. \(y^3 + 512\)
15. \(g^3 - 343\)
16. \(e^3 - 27\)
17. \(3h^9 - 192h^6\)
18. \(9n^6 - 6561n^3\)
19. \(16r^7 + 250r^4\)
20. \(135v^{11} - 1080v^8\)

ERROR ANALYSIS In Exercises 21 and 22, describe and correct the error in factoring the polynomial.

21. \(3x^3 + 27x = 3x(x^2 + 9)\)
   \[= 3x(x + 3)(x - 3)\]

22. \(x^9 + 8x^3 = (x^3)^3 + (2x)^3\)
   \[= (x^3 + 2x)[(x^3)^2 - (x^3)(2x) + (2x)^2]\]
   \[= (x^6 + 2x)(x^6 - 2x^4 + 4x^2)\]

In Exercises 23–30, factor the polynomial completely. (See Example 3.)

23. \(y^3 - 5y^2 + 6y - 30\)
24. \(m^3 - m^2 + 7m - 7\)
25. \(3a^3 + 18a^2 + 8a + 48\)
26. \(2k^3 - 20k^2 + 5k - 50\)
27. \(x^3 - 8x^2 - 4x + 32\)
28. \(z^3 - 5z^2 - 9z + 45\)
29. \(4q^3 - 16q^2 - 9q + 36\)
30. \(16n^3 + 32n^2 - n - 2\)

In Exercises 31–38, factor the polynomial completely. (See Example 4.)

31. \(49k^4 - 9\)
32. \(4m^4 - 25\)
33. \(c^4 + 9c^2 + 20\)
34. \(y^4 - 3y^2 - 28\)
35. \(16x^4 - 81\)
36. \(81x^4 - 256\)
37. \(3r^8 + 3r^5 - 60r^2\)
38. \(4n^{12} - 32n^7 + 48n^2\)

In Exercises 39–44, determine whether the binomial is a factor of \(f(x)\). (See Example 5.)

39. \(f(x) = 2x^3 + 5x^2 - 37x - 60; x - 4\)
40. \(f(x) = 3x^3 - 28x^2 + 29x + 140; x + 7\)
41. \(f(x) = 6x^5 - 15x^4 - 9x^3; x + 3\)
42. \(f(x) = 8x^5 - 58x^4 + 60x^3 + 140; x - 6\)
43. \(f(x) = 6x^4 - 6x^3 - 84x^2 + 144x; x + 4\)
44. \(f(x) = 48x^4 + 36x^3 - 138x^2 - 36x; x + 2\)
In Exercises 45–50, show that the binomial is a factor of \( f(x) \). Then factor \( f(x) \) completely. 
(See Example 6.)

45. \( f(x) = x^3 - x^2 - 20x; x + 4 \)
46. \( f(x) = x^3 - 5x^2 - 9x + 45; x - 5 \)
47. \( f(x) = x^4 - 6x^3 - 8x + 48; x - 6 \)
48. \( f(x) = x^4 + 4x^3 - 64x - 256; x + 4 \)
49. \( f(x) = x^3 - 37x + 84; x + 7 \)
50. \( f(x) = x^3 - x^2 - 24x - 36; x + 2 \)

ANALYZING RELATIONSHIPS In Exercises 51–54, match the function with the correct graph. Explain your reasoning.

51. \( f(x) = (x - 2)(x - 3)(x + 1) \)
52. \( g(x) = x(x + 2)(x + 1)(x - 2) \)
53. \( h(x) = (x + 2)(x + 3)(x - 1) \)
54. \( k(x) = x(x - 2)(x - 1)(x + 2) \)

55. MODELING WITH MATHEMATICS The volume (in cubic inches) of a packing box is modeled by \( V = 2x^3 - 19x^2 + 39x \), where \( x \) is the length (in inches). Determine the values of \( x \) for which the model makes sense. Explain your reasoning.
(See Example 7.)

56. MODELING WITH MATHEMATICS The volume (in cubic inches) of a rectangular bird cage can be modeled by \( V = 3x^3 - 17x^2 + 29x - 15 \), where \( x \) is the length (in inches). Determine the values of \( x \) for which the model makes sense. Explain your reasoning.

Using Structure In Exercises 57–64, use the method of your choice to factor the polynomial completely. Explain your reasoning.

57. \( a^6 + a^5 - 30a^4 \)
58. \( 8m^3 - 343 \)
59. \( z^3 - 7z^2 - 9z + 63 \)
60. \( 2p^8 - 12p^4 + 16p^2 \)
61. \( 64r^3 + 729 \)
62. \( 5x^5 - 10x^4 - 40x^3 \)
63. \( 16a^2 - 1 \)
64. \( 9k^3 - 24k^2 + 3k - 8 \)

65. REASONING Determine whether each polynomial is factored completely. If not, factor completely.
   a. \( 7z^4(2z^2 - z - 6) \)
   b. \((2 - n)(n^2 + 6n)(3n - 11)\)
   c. \(3(4y - 5)(9y^2 - 6y - 4)\)

66. PROBLEM SOLVING The profit \( P \) (in millions of dollars) for a T-shirt manufacturer can be modeled by \( P = -x^3 + 4x^2 + x \), where \( x \) is the number (in millions) of T-shirts produced. Currently the company produces 4 million T-shirts and makes a profit of $4 million. What lesser number of T-shirts could the company produce and still make the same profit?

67. PROBLEM SOLVING The profit \( P \) (in millions of dollars) for a shoe manufacturer can be modeled by \( P = -21x^3 + 46x \), where \( x \) is the number (in millions) of shoes produced. The company now produces 1 million shoes and makes a profit of $25 million, but it would like to cut back production. What lesser number of shoes could the company produce and still make the same profit?

Section 3.4 Factoring Polynomials 141
68. **THOUGHT PROVOKING** Find a value of \( k \) such that \( \frac{f(x)}{x - k} \) has a remainder of 0. Justify your answer.

\[
f(x) = x^3 - 3x^2 - 4x
\]

69. **COMPARING METHODS** You are taking a test where calculators are not permitted. One question asks you to evaluate \( g(7) \) for the function \( g(x) = x^3 - 7x^2 - 4x + 28 \). You use the Factor Theorem and synthetic division and your friend uses direct substitution. Whose method do you prefer? Explain your reasoning.

70. **MAKING AN ARGUMENT** You divide \( f(x) \) by \( (x - a) \) and find that the remainder does not equal 0. Your friend concludes that \( f(x) \) cannot be factored. Is your friend correct? Explain your reasoning.

71. **CRITICAL THINKING** What is the value of \( k \) such that \( x - 7 \) is a factor of \( h(x) = 2x^3 - 13x^2 - kx + 105 \)? Justify your answer.

72. **HOW DO YOU SEE IT?** Use the graph to write an equation of the cubic function in factored form. Explain your reasoning.

73. **ABSTRACT REASONING** Factor each polynomial completely.

a. \( 7ac^2 + bc^3 - 7ad^2 - bd^2 \)

b. \( x^{2n} - 2x^n + 1 \)

c. \( a^5b^2 - a^2b^4 + 2a^4b - 2ab^3 + a^3 - b^2 \)

74. **REASONING** The graph of the function \( f(x) = x^4 + 3x^3 + 2x^2 + x + 3 \) is shown. Can you use the Factor Theorem to factor \( f(x) \)? Explain.

75. **MATHEMATICAL CONNECTIONS** The standard equation of an ellipse with center \( (h, k) \) is

\[
\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1.
\]

a. An equation of an ellipse is \( 3x^2 + y^2 - 6x = 9 \). Rewrite the equation in standard form. Identify the center and then graph the ellipse.

b. Repeat part (a) using the equation \( x^2 + 2y^2 - 4y = 4 \).

c. Describe the graph of an ellipse.

76. **CRITICAL THINKING** Use the diagram to complete parts (a)–(c).

a. Explain why \( a^3 - b^3 \) is equal to the sum of the volumes of the solids I, II, and III.

b. Write an algebraic expression for the volume of each of the three solids. Leave your expressions in factored form.

c. Use the results from part (a) and part (b) to derive the factoring pattern \( a^3 - b^3 \).

---

**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

Solve the quadratic equation by factoring. *(Skills Review Handbook)*

77. \( x^2 - x - 30 = 0 \)

78. \( 2x^2 - 10x - 72 = 0 \)

79. \( 3x^2 - 11x + 10 = 0 \)

80. \( 9x^2 - 28x + 3 = 0 \)

Solve the quadratic equation by completing the square. *(Skills Review Handbook)*

81. \( x^2 - 12x + 36 = 144 \)

82. \( x^2 - 8x - 11 = 0 \)

83. \( 3x^2 + 30x + 63 = 0 \)

84. \( 4x^2 + 36x - 4 = 0 \)
3.1–3.4 What Did You Learn?

Core Vocabulary

- polynomial, p. 112
- polynomial function, p. 112
- end behavior, p. 113
- Pascal’s Triangle, p. 123
- Binomial Theorem, p. 124
- polynomial long division, p. 130
- synthetic division, p. 131
- factored completely, p. 136
- factor by grouping, p. 137
- quadratic form, p. 137

Core Concepts

Section 3.1
- Common Polynomial Functions, p. 112
- End Behavior of Polynomial Functions, p. 113
- Graphing Polynomial Functions, p. 114

Section 3.2
- Operations with Polynomials, p. 120
- Special Product Patterns, p. 121
- Pascal’s Triangle, p. 123
- The Binomial Theorem, p. 124

Section 3.3
- Polynomial Long Division, p. 130
- Synthetic Division, p. 131
- The Remainder Theorem, p. 132

Section 3.4
- Factoring Polynomials, p. 136
- Special Factoring Patterns, p. 136
- The Factor Theorem, p. 138

Mathematical Practices

1. Describe the entry points you used to analyze the function in Exercise 43 on page 118.
2. Describe how you maintained oversight in the process of factoring the polynomial in Exercise 49 on page 141.

Keeping Your Mind Focused

- When you sit down at your desk, review your notes from the last class.
- Repeat in your mind what you are writing in your notes.
- When a mathematical concept is particularly difficult, ask your teacher for another example.
3.1–3.4 Quiz

Decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient. (Section 3.1)

1. \( f(x) = 5 + 2x^2 - 3x^4 - 2x - x^3 \)  
2. \( g(x) = \frac{1}{4}x^3 + 2x - 3x^2 + 1 \)  
3. \( h(x) = 3 - 6x^3 + 4x^{-2} + 6x \)

4. Describe the \( x \)-values for which (a) \( f \) is increasing or decreasing, (b) \( f(x) > 0 \), and (c) \( f(x) < 0 \). (Section 3.1)

5. Write an expression for the area and perimeter for the figure shown. (Section 3.2)

Perform the indicated operation. (Section 3.2)

6. \( (7x^2 - 4) - (3x^2 - 5x + 1) \)
7. \( (x^2 - 3x + 2)(3x - 1) \)
8. \( (x - 1)(x + 3)(x - 4) \)

9. Use Pascal’s Triangle or the Binomial Theorem to expand \( (x + 2)^5 \). (Section 3.2)

10. Divide \( 4x^4 - 2x^3 + x^2 - 5x + 8 \) by \( x^2 - 2x - 1 \). (Section 3.3)

Factor the polynomial completely. (Section 3.4)

11. \( a^3 - 2a^2 - 8a \)
12. \( 8m^3 + 27 \)
13. \( z^3 + z^2 - 4z - 4 \)
14. \( 49b^4 - 64 \)

15. Show that \( x + 5 \) is a factor of \( f(x) = x^3 - 2x^2 - 23x + 60 \). Then factor \( f(x) \) completely. (Section 3.4)

16. The estimated price \( P \) (in cents) of stamps in the United States can be modeled by the polynomial function \( P(t) = 0.007t^3 - 0.16t^2 + 1t + 17 \), where \( t \) represents the number of years since 1990. (Section 3.1)
   a. Use a graphing calculator to graph the function for the interval \( 0 \leq t \leq 20 \).
      Describe the behavior of the graph on this interval.
   b. What was the average rate of change in the price of stamps from 1990 to 2010?

17. The volume \( V \) (in cubic feet) of a rectangular wooden crate is modeled by the function \( V(x) = 2x^3 - 11x^2 + 12x \), where \( x \) is the width (in feet) of the crate. Determine the values of \( x \) for which the model makes sense. Explain your reasoning. (Section 3.4)
3.5  Solving Polynomial Equations

Essential Question  How can you determine whether a polynomial equation has a repeated solution?

EXPLORATION 1  Cubic Equations and Repeated Solutions

Work with a partner. Some cubic equations have three distinct solutions. Others have repeated solutions. Match each cubic polynomial equation with the graph of its related polynomial function. Then solve each equation. For those equations that have repeated solutions, describe the behavior of the related function near the repeated zero using the graph or a table of values.

a. \( x^3 - 6x^2 + 12x - 8 = 0 \)
 b. \( x^3 + 3x^2 + 3x + 1 = 0 \)
 c. \( x^3 - 3x + 2 = 0 \)
 d. \( x^3 + x^2 - 2x = 0 \)
 e. \( x^3 - 3x - 2 = 0 \)
 f. \( x^3 - 3x^2 + 2x = 0 \)

A. [Graph 1]
 B. [Graph 2]
 C. [Graph 3]
 D. [Graph 4]
 E. [Graph 5]
 F. [Graph 6]

EXPLORATION 2  Quartic Equations and Repeated Solutions

Work with a partner. Determine whether each quartic equation has repeated solutions using the graph of the related quartic function or a table of values. Explain your reasoning. Then solve each equation.

a. \( x^4 - 4x^3 + 5x^2 - 2x = 0 \)
 b. \( x^4 - 2x^3 - x^2 + 2x = 0 \)
 c. \( x^4 - 4x^3 + 4x^2 = 0 \)
 d. \( x^4 + 3x^3 = 0 \)

Communicate Your Answer

3. How can you determine whether a polynomial equation has a repeated solution?

4. Write a cubic or a quartic polynomial equation that is different from the equations in Explorations 1 and 2 and has a repeated solution.
What You Will Learn

- Find solutions of polynomial equations and zeros of polynomial functions.
- Use the Rational Root Theorem.
- Use the Irrational Conjugates Theorem.

Finding Solutions and Zeros

You have used the Zero-Product Property to solve factorable quadratic equations. You can extend this technique to solve some higher-degree polynomial equations.

**EXAMPLE 1** Solving a Polynomial Equation by Factoring

Solve $2x^3 - 12x^2 + 18x = 0$.

**SOLUTION**

1. Write the equation.
2. $2x(x^2 - 6x + 9) = 0$  
3. Factor common monomial.
4. $2x(x - 3)^2 = 0$  
5. Perfect Square Trinomial Pattern
6. $2x = 0$ or $(x - 3)^2 = 0$  
7. Zero-Product Property
8. $x = 0$ or $x = 3$  
9. Solve for $x$.

The solutions, or roots, are $x = 0$ and $x = 3$.

In Example 1, the factor $x - 3$ appears more than once. This creates a **repeated solution** of $x = 3$. Note that the graph of the related function touches the $x$-axis (but does not cross the $x$-axis) at the repeated zero $x = 3$, and crosses the $x$-axis at the zero $x = 0$. This concept can be generalized as follows.

- When a factor $x - k$ of $f(x)$ is raised to an odd power, the graph of $f$ **crosses** the $x$-axis at $x = k$.
- When a factor $x - k$ of $f(x)$ is raised to an even power, the graph of $f$ **touches** the $x$-axis (but does not cross the $x$-axis) at $x = k$.

**EXAMPLE 2** Finding Zeros of a Polynomial Function

Find the zeros of $f(x) = -2x^4 + 16x^2 - 32$. Then sketch a graph of the function.

**SOLUTION**

1. $0 = -2x^4 + 16x^2 - 32$  
2. Set $f(x)$ equal to 0.
3. $0 = -2(x^4 - 8x^2 + 16)$  
5. $0 = -2(x^2 - 4)(x^2 - 4)$  
6. Factor trinomial in quadratic form.
7. $0 = -2(x + 2)(x - 2)(x + 2)(x - 2)$  
8. Difference of Two Squares Pattern
9. $0 = -2(x + 2)^2(x - 2)^2$  

Because both factors $x + 2$ and $x - 2$ are raised to an even power, the graph of $f$ touches the $x$-axis at the zeros $x = -2$ and $x = 2$.

By analyzing the original function, you can determine that the $y$-intercept is $-32$. Because the degree is even and the leading coefficient is negative, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$. Use these characteristics to sketch a graph of the function.
Monitoring Progress

Solve the equation.
1. \(4x^4 - 40x^2 + 36 = 0\)  
2. \(2x^5 + 24x = 14x^3\)

Find the zeros of the function. Then sketch a graph of the function.
3. \(f(x) = 3x^4 - 6x^2 + 3\)  
4. \(f(x) = x^3 + x^2 - 6x\)

The Rational Root Theorem
The solutions of the equation \(64x^3 + 152x^2 - 62x - 105 = 0\) are \(-\frac{5}{2}, -\frac{3}{4},\) and \(\frac{7}{8}\).
Notice that the numerators (5, 3, and 7) of the zeros are factors of the constant term, -105. Also notice that the denominators (2, 4, and 8) are factors of the leading coefficient, 64. These observations are generalized by the Rational Root Theorem.

Core Concept

The Rational Root Theorem
If \(f(x) = ax^n + \ldots + a_1x + a_0\) has integer coefficients, then every rational solution of \(f(x) = 0\) has the following form:

\[
p = \frac{\text{factor of constant term } a_0}{\text{factor of leading coefficient } a_n}
\]

The Rational Root Theorem can be a starting point for finding solutions of polynomial equations. However, the theorem lists only possible solutions. In order to find the actual solutions, you must test values from the list of possible solutions.

EXAMPLE 3 Using the Rational Root Theorem

Find all real solutions of \(x^3 - 8x^2 + 11x + 20 = 0\).

SOLUTION

The polynomial \(f(x) = x^3 - 8x^2 + 11x + 20\) is not easily factorable. Begin by using the Rational Root Theorem.

Step 1 List the possible rational solutions. The leading coefficient of \(f(x)\) is 1 and the constant term is 20. So, the possible rational solutions of \(f(x) = 0\) are

\[
x = \pm \frac{1}{1}, \pm \frac{2}{1}, \pm \frac{4}{1}, \pm \frac{5}{1}, \pm \frac{10}{1}, \pm \frac{20}{1}.
\]

Step 2 Test possible solutions using synthetic division until a solution is found.

Test \(x = 1:\)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>-8</th>
<th>11</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-7</td>
<td>4</td>
<td>24</td>
</tr>
</tbody>
</table>

\(f(1) \neq 0,\) so \(x = 1\) is not a factor of \(f(x)\).

Test \(x = -1:\)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>-8</th>
<th>11</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>1</td>
<td>-9</td>
<td>20</td>
<td>0</td>
</tr>
</tbody>
</table>

\(f(-1) = 0,\) so \(x + 1\) is a factor of \(f(x)\).

Step 3 Factor completely using the result of the synthetic division.

\[(x + 1)(x^2 - 9x + 20) = 0\]

Write as a product of factors.

\[(x + 1)(x - 4)(x - 5) = 0\]

Factor the trinomial.

So, the solutions are \(x = -1, x = 4,\) and \(x = 5.\)
In Example 3, the leading coefficient of the polynomial is 1. When the leading coefficient is not 1, the list of possible rational solutions or zeros can increase dramatically. In such cases, the search can be shortened by using a graph.

**EXAMPLE 4** Finding Zeros of a Polynomial Function

Find all real zeros of \( f(x) = 10x^4 - 11x^3 - 42x^2 + 7x + 12 \).

**SOLUTION**

**Step 1** List the possible rational zeros of \( f \): \( \pm \frac{1}{1}, \pm \frac{2}{1}, \pm \frac{3}{1}, \pm \frac{4}{1}, \pm \frac{6}{1}, \pm \frac{12}{1}, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{5}, \pm \frac{2}{5}, \pm \frac{3}{5}, \pm \frac{4}{5}, \pm \frac{6}{5}, \pm \frac{12}{5}, \pm \frac{1}{10}, \pm \frac{3}{10} \).

**Step 2** Choose reasonable values from the list above to test using the graph of the function. For \( f \), the values \( x = \frac{3}{2}, x = \frac{1}{2}, x = \frac{3}{5}, \) and \( x = \frac{12}{5} \) are reasonable based on the graph shown at the right.

**Step 3** Test the values using synthetic division until a zero is found.

\[
\begin{array}{c|ccccccc}
-\frac{3}{2} & 10 & -11 & -42 & 7 & 12 \\
 & & -15 & 39 & 9 & -69 & -4 \\
 \hline 
 & 10 & -26 & -3 & 23 & -21 & 4 \\
\end{array}
\]

\[
\begin{array}{c|ccccccc}
1 & 10 & -11 & -42 & 7 & 12 \\
 & & 10 & 8 & 17 & -12 \\
\hline 
 & 10 & 16 & -34 & 24 & 0 \\
\end{array}
\]

\(-\frac{1}{2}\) is a zero.

**Step 4** Factor out a binomial using the result of the synthetic division.

\[
f(x) = \left(x + \frac{1}{2}\right)(10x^3 - 16x^2 - 34x + 24) \quad \text{Write as a product of factors.}
\]

\[
= \left(x + \frac{1}{2}\right)(2)(5x^3 - 8x^2 - 17x + 12) \quad \text{Factor 2 out of the second factor.}
\]

\[
= (2x + 1)(5x^3 - 8x^2 - 17x + 12) \quad \text{Multiply the first factor by 2.}
\]

**Step 5** Repeat the steps above for \( g(x) = 5x^3 - 8x^2 - 17x + 12 \). Any zero of \( g \) will also be a zero of \( f \). The possible rational zeros of \( g \) are:

\[
x = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{5}, \pm \frac{2}{5}, \pm \frac{3}{5}, \pm \frac{4}{5}, \pm \frac{6}{5}, \pm \frac{12}{5}
\]

The graph of \( g \) shows that \( \frac{3}{5} \) may be a zero. Synthetic division shows that \( \frac{3}{5} \) is a zero and \( g(x) = \left(x - \frac{3}{5}\right)(5x^2 - 5x - 20) = (5x - 3)(x^2 - x - 4) \).

It follows that:

\[
f(x) = (2x + 1) \cdot g(x) = (2x + 1)(5x - 3)(x^2 - x - 4)
\]

**Step 6** Find the remaining zeros of \( f \) by solving \( x^2 - x - 4 = 0 \).

\[
x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-4)}}{2(1)} \quad \text{Substitute 1 for } a, -1 \text{ for } b, \text{ and } -4 \text{ for } c \text{ in the Quadratic Formula.}
\]

\[
x = \frac{1 \pm \sqrt{17}}{2} \quad \text{Simplify.}
\]

The real zeros of \( f \) are \( \frac{1}{2} \cdot \frac{3}{5}, \frac{1}{2} + \frac{\sqrt{17}}{2} \approx 2.56, \) and \( \frac{1 - \sqrt{17}}{2} \approx -1.56. \)
5. Find all real solutions of \( x^3 - 5x^2 - 2x + 24 = 0 \).

6. Find all real zeros of \( f(x) = 3x^4 - 2x^3 - 37x^2 + 24x + 12 \).

The Irrational Conjugates Theorem

In Example 4, notice that the irrational zeros are conjugates of the form \( a + \sqrt{b} \) and \( a - \sqrt{b} \). This illustrates the theorem below.

**Core Concept**

**The Irrational Conjugates Theorem**

Let \( f \) be a polynomial function with rational coefficients, and let \( a \) and \( b \) be rational numbers such that \( \sqrt{b} \) is irrational. If \( a + \sqrt{b} \) is a zero of \( f \), then \( a - \sqrt{b} \) is also a zero of \( f \).

**EXAMPLE 5**

Using Zeros to Write a Polynomial Function

Write a polynomial function \( f \) of least degree that has rational coefficients, a leading coefficient of 1, and the zeros \( 3 \) and \( 2 + \sqrt{5} \).

**SOLUTION**

Because the coefficients are rational and \( 2 + \sqrt{5} \) is a zero, \( 2 - \sqrt{5} \) must also be a zero by the Irrational Conjugates Theorem. Use the three zeros and the Factor Theorem to write \( f(x) \) as a product of three factors.

\[
\begin{align*}
   f(x) &= (x - 3)[x - (2 + \sqrt{5})][x - (2 - \sqrt{5})] \\
   &= (x - 3)[x - 2 - \sqrt{5}][x - 2 + \sqrt{5}] \\
   &= (x - 3)[x - 2]^2 - 5 \\
   &= (x - 3)(x^2 - 4x + 4) - 5 \\
   &= (x - 3)(x^2 - 4x + 1) \\
   &= x^3 - 4x^2 - x - 3x^2 + 12x + 3 \\
   &= x^3 - 7x^2 + 11x + 3
\end{align*}
\]

**Check**

You can check this result by evaluating \( f \) at each of its three zeros.

\[
\begin{align*}
   f(3) &= 3^3 - 7(3)^2 + 11(3) + 3 = 27 - 63 + 33 + 3 = 0 \\
   f(2 + \sqrt{5}) &= (2 + \sqrt{5})^3 - 7(2 + \sqrt{5})^2 + 11(2 + \sqrt{5}) + 3 \\
                  &= 38 + 17\sqrt{5} - 63 - 28\sqrt{5} + 22 + 11\sqrt{5} + 3 \\
                  &= 0 \\
   f(2 - \sqrt{5}) &= (2 - \sqrt{5})^3 - 7(2 - \sqrt{5})^2 + 11(2 - \sqrt{5}) + 3 \\
                  &= 0 \\
\end{align*}
\]

Because \( f(2 + \sqrt{5}) = 0 \), by the Irrational Conjugates Theorem \( f(2 - \sqrt{5}) = 0 \).

**Monitoring Progress**

7. Write a polynomial function \( f \) of least degree that has rational coefficients, a leading coefficient of 1, and the zeros \( 4 \) and \( 1 - \sqrt{5} \).
Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** If a polynomial function \( f \) has integer coefficients, then every rational solution of \( f(x) = 0 \) has the form \( \frac{p}{q} \), where \( p \) is a factor of the _______ and \( q \) is a factor of the _______.

2. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find “both” answers.

   - Find the y-intercept of the graph of \( y = x^3 - 2x^2 - x + 2 \).
   - Find the x-intercepts of the graph of \( y = x^3 - 2x^2 - x + 2 \).
   - Find all the real solutions of \( x^3 - 2x^2 - x + 2 = 0 \).
   - Find the real zeros of \( f(x) = x^3 - 2x^2 - x + 2 \).

Monitoring Progress and Modeling with Mathematics

In Exercises 3–12, solve the equation. *(See Example 1.)*

3. \( z^3 - z^2 - 12z = 0 \)
4. \( a^3 - 4a^2 + 4a = 0 \)
5. \( 2x^4 - 4x^3 = -2x^2 \)
6. \( v^3 - 2v^2 - 16v = -32 \)
7. \( 5w^3 = 50w \)
8. \( 9m^5 = 27m^3 \)
9. \( 2c^4 - 6c^3 = 12c^2 - 36c \)
10. \( p^4 + 40 = 14p^2 \)
11. \( 12n^2 + 48n = -n^3 - 64 \)
12. \( y^3 - 27 = 9y^2 - 27y \)

In Exercises 13–20, find the zeros of the function. Then sketch a graph of the function. *(See Example 2.)*

13. \( h(x) = x^3 + x^2 - 6x^2 \)
14. \( f(x) = x^4 - 18x^2 + 81 \)
15. \( p(x) = x^6 - 11x^5 + 30x^4 \)
16. \( g(x) = -2x^5 + 2x^4 + 40x^3 \)
17. \( g(x) = -4x^4 + 8x^3 + 60x^2 \)
18. \( h(x) = -x^3 - 2x^2 + 15x \)
19. \( h(x) = -x^3 - x^2 + 9x + 9 \)
20. \( p(x) = x^3 - 5x^2 - 4x + 20 \)

**3.5 Exercises**

21. **USING EQUATIONS** According to the Rational Root Theorem, which is *not* a possible solution of the equation \( 2x^4 - 5x^3 + 10x^2 - 9 = 0 \)?
   
   \( \text{A) } -9 \quad \text{B) } -\frac{1}{2} \quad \text{C) } \frac{5}{2} \quad \text{D) } 3 \)

22. **USING EQUATIONS** According to the Rational Root Theorem, which is *not* a possible zero of the function \( f(x) = 40x^5 - 42x^4 - 107x^3 + 107x^2 + 33x - 36 \)?
   
   \( \text{A) } -\frac{2}{3} \quad \text{B) } -\frac{3}{8} \quad \text{C) } \frac{1}{4} \quad \text{D) } \frac{4}{5} \)

**ERROR ANALYSIS** In Exercises 23 and 24, describe and correct the error in listing the possible rational zeros of the function.

23. \( f(x) = x^3 + 5x^2 - 9x - 45 \)
   
   Possible rational zeros of \( f \): \( 1, 3, 5, 9, 15, 45 \)

24. \( f(x) = 3x^3 + 13x^2 - 41x + 8 \)
   
   Possible rational zeros of \( f \): \( \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm \frac{1}{3}, \pm \frac{3}{2}, \pm \frac{3}{4}, \pm \frac{3}{8} \)

In Exercises 25–32, find all the real solutions of the equation. *(See Example 3.)*

25. \( x^3 + x^2 - 17x + 15 = 0 \)
26. \( x^3 - 2x^2 - 5x + 6 = 0 \)
27. \( x^3 - 10x^2 + 19x + 30 = 0 \)
28. \( x^3 + 4x^2 - 11x - 30 = 0 \)
29. \( x^3 - 6x^2 - 7x + 60 = 0 \)
30. \( x^3 - 16x^2 + 55x + 72 = 0 \)
31. \( 2x^3 - 3x^2 - 50x - 24 = 0 \)
32. \( 3x^3 + x^2 - 38x + 24 = 0 \)

In Exercises 33–38, find all the real zeros of the function. (See Example 4.)
33. \( f(x) = x^3 - 2x^2 - 23x + 60 \)
34. \( g(x) = x^3 - 28x - 48 \)
35. \( h(x) = x^3 + 10x^2 + 31x + 30 \)
36. \( f(x) = x^3 - 14x^2 + 55x - 42 \)
37. \( p(x) = 2x^3 - x^2 - 27x + 36 \)
38. \( g(x) = 3x^3 - 25x^2 + 58x - 40 \)

Using Tools In Exercises 39 and 40, use the graph to shorten the list of possible rational zeros of the function. Then find all real zeros of the function.
39. \( f(x) = 4x^3 - 20x + 16 \)  
40. \( f(x) = 4x^3 - 49x - 60 \)

In Exercises 41–46, write a polynomial function \( f \) of least degree that has a leading coefficient of 1 and the given zeros. (See Example 5.)
41. \(-2, 3, 6\)  
42. \(-4, -2, 5\)  
43. \(-2, 1 + \sqrt{7}\)  
44. \(4, 6 - \sqrt{7}\)  
45. \(-6, 0, 3 - \sqrt{5}\)  
46. \(0, 5, -5 + \sqrt{8}\)

Comparing Methods Solve the equation \( x^3 - 4x^2 - 9x + 36 = 0 \) using two different methods. Which method do you prefer? Explain your reasoning.

Reasoning Is it possible for a cubic function to have more than three real zeros? Explain.

49. Problem Solving At a factory, molten glass is poured into molds to make paperweights. Each mold is a rectangular prism with a height 3 centimeters greater than the length of each side of its square base. Each mold holds 112 cubic centimeters of glass. What are the dimensions of the mold?

50. Mathematical Connections The volume of the cube shown is 8 cubic centimeters.
   a. Write a polynomial equation that you can use to find the value of \( x \).
   b. Identify the possible rational solutions of the equation in part (a).
   c. Use synthetic division to find a rational solution of the equation. Show that no other real solutions exist.
   d. What are the dimensions of the cube?

51. Problem Solving Archaeologists discovered a huge hydraulic concrete block at the ruins of Caesarea with a volume of 945 cubic meters. The block is \( x \) meters high by 12\(x - 15\) meters long by \( 12x - 21\) meters wide. What are the dimensions of the block?

52. Making an Argument Your friend claims that when a polynomial function has a leading coefficient of 1 and the coefficients are all integers, every possible rational zero is an integer. Is your friend correct? Explain your reasoning.

53. Modeling With Mathematics During a 10-year period, the amount (in millions of dollars) of athletic equipment \( E \) sold domestically can be modeled by \( E(t) = -20r^3 + 252r^2 - 280r + 21,614 \), where \( t \) is in years.
   a. Write a polynomial equation to find the year when about \$24,014,000,000\ of athletic equipment is sold.
   b. List the possible whole-number solutions of the equation in part (a). Consider the domain when making your list of possible solutions.
   c. Use synthetic division to find when \$24,014,000,000\ of athletic equipment is sold.
54. **THOUGHT PROVOKING** Write a third or fourth degree polynomial function that has zeros at \( \pm \frac{1}{2} \). Justify your answer.

55. **MODELING WITH MATHEMATICS** You are designing a marble basin that will hold a fountain for a city park. The sides and bottom of the basin should be 1 foot thick. Its outer length should be twice its outer width and outer height. What should the outer dimensions of the basin be if it is to hold 36 cubic feet of water?

56. **HOW DO YOU SEE IT?** Use the information in the graph to answer the questions.

   - What are the real zeros of the function \( f \)?
   - Write an equation of the quartic function in factored form.

57. **REASONING** Determine the value of \( k \) for each equation so that the given \( x \)-value is a solution.
   
   - \( x^3 - 6x^2 - 7x + k = 0; x = 4 \)
   - \( 2x^3 + 7x^2 - kx - 18 = 0; x = -6 \)
   - \( kx^3 - 35x^2 + 19x + 30 = 0; x = 5 \)

58. **WRITING EQUATIONS** Write a polynomial function \( g \) of least degree that has rational coefficients, a leading coefficient of 1, and the zeros \( -2 + \sqrt{7} \) and \( 3 + \sqrt{2} \).

In Exercises 59–62, solve \( f(x) = g(x) \) by graphing and algebraic methods.

59. \( f(x) = x^3 + x^2 - x - 1; g(x) = -x + 1 \)
60. \( f(x) = x^4 - 5x^3 + 2x^2 + 8x; g(x) = -x^2 + 6x - 8 \)
61. \( f(x) = x^3 - 4x^2 + 4x; g(x) = -2x + 4 \)
62. \( f(x) = x^4 + 2x^3 - 11x^2 - 12x + 36; g(x) = -x^2 - 6x - 9 \)

63. **MODELING WITH MATHEMATICS** You are building a pair of ramps for a loading platform. The left ramp is twice as long as the right ramp. If 150 cubic feet of concrete are used to build the ramps, what are the dimensions of each ramp?

64. **MODELING WITH MATHEMATICS** Some ice sculptures are made by filling a mold and then freezing it. You are making an ice mold for a school dance. It is to be shaped like a pyramid with a height 1 foot greater than the length of each side of its square base. The volume of the ice sculpture is 4 cubic feet. What are the dimensions of the mold?

65. **ABSTRACT REASONING** Let \( a_n \) be the leading coefficient of a polynomial function \( f \) and \( a_0 \) be the constant term. If \( a_n \) has \( r \) factors and \( a_0 \) has \( s \) factors, what is the greatest number of possible rational zeros of \( f \) that can be generated by the Rational Zero Theorem? Explain your reasoning.

---

**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

66. \( h(x) = -3x^2 + 2x - 9 + \sqrt{4}x^3 \)
67. \( g(x) = 2x^3 - 7x^2 - 3x^{-1} + x \)
68. \( f(x) = \frac{1}{3}x^2 + 2x^3 - 4x^4 - \sqrt{3} \)
69. \( p(x) = 2x - 5x^3 + 9x^2 + \sqrt[3]{x} + 1 \)

Find the zeros of the function. **(Skills Review Handbook)**

70. \( f(x) = 7x^2 + 42 \)
71. \( g(x) = 9x^2 + 81 \)
72. \( h(x) = 5x^2 + 40 \)
73. \( f(x) = 8x^2 - 1 \)
3.6 The Fundamental Theorem of Algebra

Essential Question How can you determine whether a polynomial equation has imaginary solutions?

EXPLORATION 1 Cubic Equations and Imaginary Solutions

Work with a partner. Match each cubic polynomial equation with the graph of its related polynomial function. Then find all solutions. Make a conjecture about how you can use a graph or table of values to determine the number and types of solutions of a cubic polynomial equation.

a. \(x^3 - 3x^2 + x + 5 = 0\)

b. \(x^3 - 2x^2 - x + 2 = 0\)

c. \(x^3 - x^2 - 4x + 4 = 0\)

d. \(x^3 + 5x^2 + 8x + 6 = 0\)

e. \(x^3 - 3x^2 + x - 3 = 0\)

f. \(x^3 - 3x^2 + 2x = 0\)

EXPLORATION 2 Quartic Equations and Imaginary Solutions

Work with a partner. Use the graph of the related quartic function, or a table of values, to determine whether each quartic equation has imaginary solutions. Explain your reasoning. Then find all solutions.

a. \(x^4 - 2x^3 - x^2 + 2x = 0\)

b. \(x^4 - 1 = 0\)

c. \(x^4 + x^3 - x - 1 = 0\)

d. \(x^4 - 3x^3 + x^2 + 3x - 2 = 0\)

Communicate Your Answer

3. How can you determine whether a polynomial equation has imaginary solutions?

4. Is it possible for a cubic equation to have three imaginary solutions? Explain your reasoning.
3.6 Lesson

What You Will Learn

- Use the Fundamental Theorem of Algebra.
- Find conjugate pairs of complex zeros of polynomial functions.
- Use Descartes’s Rule of Signs.

Core Vocabulary

complex conjugates, p. 155

Previous

repeated solution
degree of a polynomial
solution of an equation
zero of a function conjugates

The Fundamental Theorem of Algebra

The table shows several polynomial equations and their solutions, including repeated solutions. In the last equation, the repeated solution \( x = -1 \) is counted twice.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Degree</th>
<th>Solution(s)</th>
<th>Number of solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2x - 1 = 0 )</td>
<td>1</td>
<td>( \frac{1}{2} )</td>
<td>1</td>
</tr>
<tr>
<td>( x^2 + 2 = 0 )</td>
<td>2</td>
<td>( \pm \sqrt{2} )</td>
<td>2</td>
</tr>
<tr>
<td>( x^3 + x^2 - x - 1 = 0 )</td>
<td>3</td>
<td>(-1, -1, 1)</td>
<td>3</td>
</tr>
</tbody>
</table>

In the table, note the relationship between the degree of the polynomial \( f(x) \) and the number of solutions of \( f(x) = 0 \). This relationship is generalized by the Fundamental Theorem of Algebra.

Core Concept

The Fundamental Theorem of Algebra

**Theorem** If \( f(x) \) is a polynomial of degree \( n \) where \( n > 0 \), then the equation \( f(x) = 0 \) has at least one solution in the set of complex numbers.

**Corollary** If \( f(x) \) is a polynomial of degree \( n \) where \( n > 0 \), then the equation \( f(x) = 0 \) has exactly \( n \) solutions provided each solution repeated twice is counted as two solutions, each solution repeated three times is counted as three solutions, and so on.

STUDY TIP

The statements “the polynomial equation \( f(x) = 0 \) has exactly \( n \) solutions” and “the polynomial function \( f \) has exactly \( n \) zeros” are equivalent.

EXAMPLE 1 Solving a Polynomial Equation

How many solutions does \( x^4 + x^3 + 8x + 8 = 0 \) have? Find all the solutions.

**SOLUTION**

Because \( x^4 + x^3 + 8x + 8 = 0 \) is a polynomial equation of degree 4, it has four solutions. Notice that you can use factoring by grouping to begin solving the equation.

\[
(x^4 + x^3) + (8x + 8) = 0 \quad \text{Group terms with common factors.}
\]
\[
x^3(x + 1) + 8(x + 1) = 0 \quad \text{Factor out GCF of each pair of terms.}
\]
\[
(x + 1)(x^3 + 8) = 0 \quad \text{Factor out } (x + 1).\]
\[
(x + 1)(x + 2)(x^2 - 2x + 4) = 0 \quad \text{Sum of Two Cubes Pattern}
\]

The linear factors indicate that \(-2\) and \(-1\) are solutions. To find the remaining two solutions, solve \( x^2 - 2x + 4 = 0 \) by using the Quadratic Formula.

\[
x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)} = 1 \pm i\sqrt{3}
\]

\( \text{The solutions are } -2, -1, 1 - i\sqrt{3}, \text{ and } 1 + i\sqrt{3} \).
EXAMPLE 2  Finding the Zeros of a Polynomial Function

Find all zeros of \( f(x) = x^5 + x^3 - 2x^2 - 12x - 8 \).

SOLUTION

Step 1  Find the rational zeros of \( f \). Because \( f \) is a polynomial function of degree 5, it has five zeros. The possible rational zeros are \( \pm 1, \pm 2, \pm 4, \) and \( \pm 8 \). Using synthetic division, you can determine that \(-1\) is a zero repeated twice and \(2\) is also a zero.

Step 2  Write \( f(x) \) in factored form. Dividing \( f(x) \) by its known factors \( x + 1, x + 1, \) and \( x - 2 \) gives a quotient of \( x^2 + 4 \). So,

\[ f(x) = (x + 1)^2(x - 2)(x^2 + 4) \]

Step 3  Find the complex zeros of \( f \). Solving \( x^2 + 4 = 0 \), you get \( x = \pm 2i \). This means \( x^2 + 4 = (x + 2i)(x - 2i) \).

\[ f(x) = (x + 1)^2(x - 2)(x + 2i)(x - 2i) \]

From the factorization, there are five zeros. The zeros of \( f \) are 

\(-1, -1, 2, -2i, \) and \(2i\).

The graph of \( f \) and the real zeros are shown. Notice that only the real zeros appear as \( x \)-intercepts. Also, the graph of \( f \) touches the \( x \)-axis at the repeated zero \( x = -1 \) and crosses the \( x \)-axis at \( x = 2 \).

Monitoring Progress

Help in English and Spanish at BigIdeasMath.com

Identify the number of solutions of the polynomial equation. Then find all solutions of the equation.

1. \( x^5 - 4x^3 - x^2 + 4 = 0 \)
2. \( x^4 + 7x^2 - 144 = 0 \)

Find all zeros of the polynomial function.

3. \( f(x) = x^3 + 7x^2 + 16x + 12 \)
4. \( f(x) = x^5 - 3x^4 + 5x^3 - x^2 - 6x + 4 \)

Complex Conjugates

Pairs of complex numbers of the forms \( a + bi \) and \( a - bi \), where \( b \neq 0 \), are called complex conjugates. In Example 2, notice that the zeros \( 2i \) and \(-2i \) are complex conjugates. This illustrates the next theorem.

The Complex Conjugates Theorem

If \( f \) is a polynomial function with real coefficients, and \( a + bi \) is an imaginary zero of \( f \), then \( a - bi \) is also a zero of \( f \).
Example 3  Using Zeros to Write a Polynomial Function

Write a polynomial function \( f \) of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 2 and \( 3 + i \).

**Solution**

Because the coefficients are rational and \( 3 + i \) is a zero, \( 3 - i \) must also be a zero by the Complex Conjugates Theorem. Use the three zeros and the Factor Theorem to write \( f(x) \) as a product of three factors.

\[
 f(x) = (x - 2)(x - (3 + i))(x - (3 - i))
\]

Write \( f(x) \) in factored form.

\[
 f(x) = (x - 2)((x - 3) - i)((x - 3) + i)
\]

Regroup terms.

\[
 f(x) = (x - 2)((x - 3)^2 - i^2)
\]

Multiply.

\[
 f(x) = (x - 2)((x^2 - 6x + 9) - (-1))
\]

Expand binomial and use \( i^2 = -1 \).

\[
 f(x) = (x - 2)(x^2 - 6x + 10)
\]

Simplify.

\[
 f(x) = x^3 - 6x^2 + 10x - 2x^2 + 22x - 20
\]

Multiply.

\[
 f(x) = x^3 - 8x^2 + 22x - 20
\]

Combine like terms.

**Check**

You can check this result by evaluating \( f \) at each of its three zeros.

\[
 f(2) = (2)^3 - 8(2)^2 + 22(2) - 20 = 8 - 32 + 44 - 20 = 0 \checkmark
\]

\[
 f(3 + i) = (3 + i)^3 - 8(3 + i)^2 + 22(3 + i) - 20
\]

Expanded binomial and use \( i^2 = -1 \),

\[
 = 18 + 26i - 64 - 48i + 66 + 22i - 20
\]

Simplify.

\[
 = 0 \checkmark
\]

Because \( f(3 + i) = 0 \), by the Complex Conjugates Theorem \( f(3 - i) = 0 \). \checkmark

Monitoring Progress

Write a polynomial function \( f \) of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros.

5. \(-1, 4i\)  6. \(3, 1 + i\sqrt{5}\)  7. \(\sqrt{2}, 1 - 3i\)  8. \(2, 2i, 4 - \sqrt{6}\)

**Descartes’s Rule of Signs**

French mathematician René Descartes (1596–1650) found the following relationship between the coefficients of a polynomial function and the number of positive and negative zeros of the function.

**Core Concept**

**Descartes’s Rule of Signs**

Let \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 \) be a polynomial function with real coefficients.

- The number of positive real zeros of \( f \) is equal to the number of changes in sign of the coefficients of \( f(x) \) or is less than this by an even number.
- The number of negative real zeros of \( f \) is equal to the number of changes in sign of the coefficients of \( f(-x) \) or is less than this by an even number.
### Example 4 Using Descartes’s Rule of Signs

Determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for \( f(x) = x^6 - 2x^5 + 3x^4 - 10x^3 - 6x^2 - 8x - 8 \).

**SOLUTION**

\[ f(x) = x^6 - 2x^5 + 3x^4 - 10x^3 - 6x^2 - 8x - 8. \]

The coefficients in \( f(x) \) have 3 sign changes, so \( f \) has 3 or 1 positive real zero(s).

\[ f(-x) = (-x)^6 - 2(-x)^5 + 3(-x)^4 - 10(-x)^3 - 6(-x)^2 - 8(-x) - 8 \]

\[ = x^6 + 2x^5 + 3x^4 + 10x^3 - 6x^2 + 8x - 8 \]

The coefficients in \( f(-x) \) have 3 sign changes, so \( f \) has 3 or 1 negative zero(s).

The possible numbers of zeros for \( f \) are summarized in the table below.

<table>
<thead>
<tr>
<th>Positive real zeros</th>
<th>Negative real zeros</th>
<th>Imaginary zeros</th>
<th>Total zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

### Example 5

**Real-Life Application**

A tachometer measures the speed (in revolutions per minute, or RPMs) at which an engine shaft rotates. For a certain boat, the speed \( s \) (in hundreds of RPMs) of the engine shaft and the speed \( s \) (in miles per hour) of the boat are modeled by

\[ s(x) = 0.00547x^3 - 0.225x^2 + 3.62x - 11.0. \]

What is the tachometer reading when the boat travels 15 miles per hour?

**SOLUTION**

Substitute 15 for \( s(x) \) in the function. You can rewrite the resulting equation as

\[ 0 = 0.00547x^3 - 0.225x^2 + 3.62x - 26.0. \]

The related function to this equation is \( f(x) = 0.00547x^3 - 0.225x^2 + 3.62x - 26.0 \). By Descartes’s Rule of Signs, you know \( f \) has 3 or 1 positive real zero(s). In the context of speed, negative real zeros and imaginary zeros do not make sense, so you do not need to check for them. To approximate the positive real zeros of \( f \), use a graphing calculator.

From the graph, there is 1 real zero, \( x \approx 19.9 \).

The tachometer reading is about 1990 RPMs.

### Monitoring Progress

Determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for the function.

9. \( f(x) = x^3 + 9x - 25 \)
10. \( f(x) = 3x^4 - 7x^3 + x^2 - 13x + 8 \)
11. **What If?** In Example 5, what is the tachometer reading when the boat travels 20 miles per hour?
3.6 Exercises

Vocabulary and Core Concept Check

1. COMPLETE THE SENTENCE The expressions $5 + i$ and $5 - i$ are ____________.

2. WRITING How many solutions does the polynomial equation $(x + 8)(x - 1) = 0$ have? Explain.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, identify the number of solutions of the polynomial equation. Then find all solutions of the equation. (See Example 1.)

3. $x^3 + 64 = 0$
4. $8y^4 - y = 0$
5. $r^6 - r^2 = 0$
6. $z^4 + 5z^2 - 14 = 0$
7. $s^5 - s^3 - s^2 + 1 = 0$
8. $x^3 - 3x^2 + 2x - 6 = 0$

In Exercises 9–16, find all zeros of the polynomial function. (See Example 2.)

9. $f(x) = x^4 - 6x^3 + 7x^2 + 6x - 8$
10. $f(x) = x^4 + 5x^3 - 7x^2 - 29x + 30$
11. $g(x) = x^4 - 9x^2 - 4x + 12$
12. $h(x) = x^3 + 5x^2 - 4x - 20$
13. $g(x) = x^4 + 4x^3 + 7x^2 + 16x + 12$
14. $h(x) = x^4 - x^3 + 7x^2 - 9x - 18$
15. $g(x) = x^5 + 3x^4 - x^3 - 2x^2 - 12x - 16$
16. $f(x) = x^5 - 20x^3 + 20x^2 - 21x + 20$

ANALYZING RELATIONSHIPS In Exercises 17–20, determine the number of imaginary zeros for the function with the given degree and graph. Explain your reasoning.

17. Degree: 4
18. Degree: 5

19. Degree: 2
20. Degree: 3

In Exercises 21–28, write a polynomial function $f$ of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros. (See Example 3.)

21. $-5, -1, 2$
22. $-2, 1, 3$
23. $3, 4 + i$
24. $2, 5 - i$
25. $4, -\sqrt{5}$
26. $3i, 2 - i$
27. $2, 1 + i, 2 - \sqrt{3}$
28. $3, 4 + 2i, 1 + \sqrt{7}$

ERROR ANALYSIS In Exercises 29 and 30, describe and correct the error in writing a polynomial function with rational coefficients and the given zero(s).

29. Zeros: $2, 1 + i$

30. Zero: $2 + i$
31. **OPEN-ENDED** Write a polynomial function of degree 6 with zeros 1, 2, and −i. Justify your answer.

32. **REASONING** Two zeros of \( f(x) = x^3 - 6x^2 - 16x + 96 \) are 4 and −4. Explain why the third zero must also be a real number.

In Exercises 33–40, determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for the function. (See Example 4.)

33. \( g(x) = x^4 - x^3 - 6 \)

34. \( g(x) = -x^3 + 5x^2 + 12 \)

35. \( g(x) = x^3 - 4x^2 + 8x + 7 \)

36. \( g(x) = x^5 - 2x^3 - x^2 + 6 \)

37. \( g(x) = x^5 - 3x^3 + 8x - 10 \)

38. \( g(x) = x^5 + 7x^4 - 4x^3 - 3x^2 + 9x - 15 \)

39. \( g(x) = x^6 + x^5 - 3x^4 + x^3 + 5x^2 + 9x - 18 \)

40. \( g(x) = x^7 + 4x^4 - 10x + 25 \)

41. **REASONING** Which is not a possible classification of zeros for \( f(x) = x^3 - 4x^2 + 6x^2 + 2x - 6 \)? Explain.
   - A three positive real zeros, two negative real zeros, and zero imaginary zeros
   - B three positive real zeros, zero negative real zeros, and two imaginary zeros
   - C one positive real zero, four negative real zeros, and zero imaginary zeros
   - D one positive real zero, two negative real zeros, and two imaginary zeros

42. **USING STRUCTURE** Use Descartes's Rule of Signs to determine which function has at least 1 positive real zero.
   - A \( f(x) = x^4 + 2x^2 - 9x^2 - 2x + 8 \)
   - B \( f(x) = x^4 + 4x^3 + 8x^2 + 16x + 16 \)
   - C \( f(x) = -x^4 - 5x^3 - 4 \)
   - D \( f(x) = x^4 + 4x^3 + 7x^2 + 12x + 12 \)

43. **MODELING WITH MATHEMATICS** From 1890 to 2000, the American Indian, Eskimo, and Aleut population \( P \) (in thousands) can be modeled by the function \( P = 0.0024t^2 + 0.24t^2 + 49t + 243 \), where \( t \) is the number of years since 1890. In which year did the population first reach 722,000? (See Example 5.)

44. **MODELING WITH MATHEMATICS** Over a period of 14 years, the number \( N \) of inland lakes infested with zebra mussels in a certain state can be modeled by \( N = -0.0284t^4 + 0.5937t^3 - 2.464t^2 + 8.33t - 2.5 \) where \( t \) is time (in years). In which year did the number of infested inland lakes first reach 120?

45. **MODELING WITH MATHEMATICS** For the 12 years that a grocery store has been open, its annual revenue \( R \) (in millions of dollars) can be modeled by the function \( R = 0.0001(-t^4 + 12t^3 - 77t^2 + 600t + 13,650) \) where \( t \) is the number of years since the store opened. In which year(s) was the revenue $1.5 million?

46. **MAKING AN ARGUMENT** Your friend claims that \( 2 - i \) is a complex zero of the polynomial function \( f(x) = x^3 - 2x^2 + 2x + 5i \), but that its conjugate is not a zero. You claim that both \( 2 - i \) and its conjugate must be zeros by the Complex Conjugates Theorem. Who is correct? Justify your answer.

47. **MATHEMATICAL CONNECTIONS** A solid monument with the dimensions shown is to be built using 1000 cubic feet of marble. What is the value of \( x \)?
48. **THOUGHT PROVOKING** Write and graph a polynomial function of degree 5 that has all positive or negative real zeros. Label each x-intercept. Then write the function in standard form.

49. **WRITING** The graph of the constant polynomial function \( f(x) = 2 \) is a line that does not have any x-intercepts. Does the function contradict the Fundamental Theorem of Algebra? Explain.

50. **HOW DO YOU SEE IT?** The graph represents a polynomial function of degree 6.

(a) How many positive real zeros does the function have? negative real zeros? imaginary zeros?

(b) Use Descartes’s Rule of Signs and your answers in part (a) to describe the possible sign changes in the coefficients of \( f(x) \).

51. **FINDING A PATTERN** Use a graphing calculator to graph the function \( f(x) = (x + 3)^n \) for \( n = 2, 3, 4, 5, 6, \) and 7.

(a) Compare the graphs when \( n \) is even and \( n \) is odd.

(b) Describe the behavior of the graph near the zero \( x = -3 \) as \( n \) increases.

(c) Use your results from parts (a) and (b) to describe the behavior of the graph of \( g(x) = (x - 4)^2 \) near \( x = 4 \).

52. **DRAWING CONCLUSIONS** Find the zeros of each function.

\[
\begin{align*}
  f(x) &= x^2 - 5x + 6 \\
  g(x) &= x^3 - 7x + 6 \\
  h(x) &= x^4 + 2x^3 + x^2 + 8x - 12 \\
  k(x) &= x^5 - 3x^4 - 9x^3 + 25x^2 - 6x 
\end{align*}
\]

(a) Describe the relationship between the sum of the zeros of a polynomial function and the coefficients of the polynomial function.

(b) Describe the relationship between the product of the zeros of a polynomial function and the coefficients of the polynomial function.

53. **PROBLEM SOLVING** You want to save money so you can buy a used car in four years. At the end of each summer, you deposit \$1000\) earned from summer jobs into your bank account. The table shows the value of your deposits over the four-year period. In the table, \( g \) is the growth factor \( 1 + r \), where \( r \) is the annual interest rate expressed as a decimal.

<table>
<thead>
<tr>
<th>Deposit</th>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
<th>Year 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st Deposit</td>
<td>1000</td>
<td>1000g</td>
<td>1000g^2</td>
<td>1000g^3</td>
</tr>
<tr>
<td>2nd Deposit</td>
<td>—</td>
<td>1000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3rd Deposit</td>
<td>—</td>
<td>—</td>
<td>1000</td>
<td></td>
</tr>
<tr>
<td>4th Deposit</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>1000</td>
</tr>
</tbody>
</table>

(a) Copy and complete the table.

(b) Write a polynomial function that gives the value \( v \) of your account at the end of the fourth summer in terms of \( g \).

(c) You want to buy a car that costs about \$4300. What growth factor do you need to obtain this amount? What annual interest rate do you need?

---

**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

Describe the transformation of \( f(x) = x^2 \) represented by \( g \). Then graph each function. (Section 2.5)

54. \( g(x) = -3x^2 \)

55. \( g(x) = (x - 4)^2 + 6 \)

56. \( g(x) = -(x - 1)^2 \)

57. \( g(x) = 5(x + 4)^2 \)

Write a function \( g \) whose graph represents the indicated transformation of the graph of \( f \). (Sections 2.2 and 2.5)

58. \( f(x) = x \); vertical shrink by a factor of \( \frac{1}{3} \) and a reflection in the y-axis

59. \( f(x) = |x + 1| - 3 \); horizontal stretch by a factor of 9

60. \( f(x) = x^2 \); reflection in the x-axis, followed by a translation 2 units right and 7 units up

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3.7 Transformations of Polynomial Functions

**Essential Question** How can you transform the graph of a polynomial function?

**EXPLORATION 1** Transforming the Graph of a Cubic Function

**Work with a partner.** The graph of the cubic function

\[ f(x) = x^3 \]

is shown. The graph of each cubic function \( g \) represents a transformation of the graph of \( f \). Write a rule for \( g \). Use a graphing calculator to verify your answers.

a. 

![Graph of g](image)

b. 

![Graph of g](image)

c. 

![Graph of g](image)

d. 

![Graph of g](image)

**EXPLORATION 2** Transforming the Graph of a Quartic Function

**Work with a partner.** The graph of the quartic function

\[ f(x) = x^4 \]

is shown. The graph of each quartic function \( g \) represents a transformation of the graph of \( f \). Write a rule for \( g \). Use a graphing calculator to verify your answers.

a. 

![Graph of g](image)

b. 

![Graph of g](image)

**LOOKING FOR STRUCTURE**

To be proficient in math, you need to see complicated things, such as some algebraic expressions, as being single objects or as being composed of several objects.

**Communicate Your Answer**

3. How can you transform the graph of a polynomial function?

4. Describe the transformation of \( f(x) = x^4 \) represented by \( g(x) = (x + 1)^4 + 3 \). Then graph \( g \).

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3.7 Lesson

What You Will Learn

- Describe transformations of polynomial functions.
- Write transformations of polynomial functions.

Describing Transformations of Polynomial Functions

You can transform graphs of polynomial functions in the same way you transformed graphs of linear functions, absolute value functions, and quadratic functions. Examples of transformations of the graph of \( f(x) = x^4 \) are shown below.

Core Concept

<table>
<thead>
<tr>
<th>Transformation</th>
<th>( f(x) ) Notation</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Horizontal Translation</strong></td>
<td>( f(x - h) )</td>
<td>( g(x) = (x - 5)^4 )</td>
</tr>
<tr>
<td>Graph shifts left or right.</td>
<td></td>
<td>( g(x) = (x + 2)^4 )</td>
</tr>
<tr>
<td><strong>Vertical Translation</strong></td>
<td>( f(x) + k )</td>
<td>( g(x) = x^4 + 1 )</td>
</tr>
<tr>
<td>Graph shifts up or down.</td>
<td></td>
<td>( g(x) = x^4 - 4 )</td>
</tr>
<tr>
<td><strong>Reflection</strong></td>
<td>( f(-x) )</td>
<td>( g(x) = (-x)^4 = x^4 )</td>
</tr>
<tr>
<td>Graph flips over ( x )- or ( y )-axis.</td>
<td>( -f(x) )</td>
<td>( g(x) = -x^4 )</td>
</tr>
<tr>
<td><strong>Horizontal Stretch or Shrink</strong></td>
<td>( f(ax) )</td>
<td>( g(x) = (2x)^4 )</td>
</tr>
<tr>
<td>Graph stretches away from or shrinks toward ( y )-axis.</td>
<td></td>
<td>( g(x) = \left(\frac{1}{2}x\right)^4 )</td>
</tr>
<tr>
<td><strong>Vertical Stretch or Shrink</strong></td>
<td>( a \cdot f(x) )</td>
<td>( g(x) = 8x^4 )</td>
</tr>
<tr>
<td>Graph stretches away from or shrinks toward ( x )-axis.</td>
<td></td>
<td>( g(x) = \frac{1}{2}x^4 )</td>
</tr>
</tbody>
</table>

**EXAMPLE 1** Translating a Polynomial Function

Describe the transformation of \( f(x) = x^3 \) represented by \( g(x) = (x + 5)^3 + 2 \). Then graph each function.

**SOLUTION**

Notice that the function is of the form \( g(x) = (x - h)^3 + k \). Rewrite the function to identify \( h \) and \( k \).

\[
g(x) = (x - (-5))^3 + 2
\]

Because \( h = -5 \) and \( k = 2 \), the graph of \( g \) is a translation 5 units left and 2 units up of the graph of \( f \).

**Monitoring Progress**

1. Describe the transformation of \( f(x) = x^4 \) represented by \( g(x) = (x - 3)^4 - 1 \). Then graph each function.
EXAMPLE 2  Transforming Polynomial Functions

Describe the transformation of \( f \) represented by \( g \). Then graph each function.

a. \( f(x) = x^4, g(x) = -\frac{1}{4}x^4 \)

SOLUTION

a. Notice that the function is of the form \( g(x) = -ax^4 \), where \( a = \frac{1}{4} \).

So, the graph of \( g \) is a reflection in the \( x \)-axis and a vertical shrink by a factor of \( \frac{1}{4} \) of the graph of \( f \).

b. \( f(x) = x^5, g(x) = (2x)^5 - 3 \)

b. Notice that the function is of the form \( g(x) = (ax)^5 + k \), where \( a = 2 \) and \( k = -3 \).

So, the graph of \( g \) is a horizontal shrink by a factor of \( \frac{1}{2} \) and a translation 3 units down of the graph of \( f \).

Monitoring Progress

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2. Describe the transformation of \( f(x) = x^3 \) represented by \( g(x) = 4(x + 2)^3 \). Then graph each function.

Writing Transformations of Polynomial Functions

EXAMPLE 3  Writing Transformed Polynomial Functions

Let \( f(x) = x^3 + x^2 + 1 \). Write a rule for \( g \) and then graph each function. Describe the graph of \( g \) as a transformation of the graph of \( f \).

a. \( g(x) = f(-x) \)

SOLUTION

a. \( g(x) = f(-x) \)

\[ \begin{align*}
  &= (-x)^3 + (-x)^2 + 1 \\
  &= -x^3 + x^2 + 1 \\
\end{align*} \]

So, the graph of \( g \) is a reflection in the \( y \)-axis of the graph of \( f \).

b. \( g(x) = 3f(x) \)

b. \( g(x) = 3f(x) \)

\[ \begin{align*}
  &= 3(x^3 + x^2 + 1) \\
  &= 3x^3 + 3x^2 + 3 \\
\end{align*} \]

So, the graph of \( g \) is a vertical stretch by a factor of 3 of the graph of \( f \).

REMEMBER

Vertical stretches and shrinks do not change the \( x \)-intercept(s) of a graph. You can observe this using the graph in Example 3(b).
EXAMPLE 4  Writing a Transformed Polynomial Function

Let the graph of \( g \) be a vertical stretch by a factor of 2, followed by a translation 3 units up of the graph of \( f(x) = x^4 - 2x^2 \). Write a rule for \( g \).

SOLUTION

Step 1  First write a function \( h \) that represents the vertical stretch of \( f \).

\[
h(x) = 2 \cdot f(x)
\]

Multiply the output by 2.

\[
= 2(x^4 - 2x^2)
\]

Substitute \( x^4 - 2x^2 \) for \( f(x) \).

\[
= 2x^4 - 4x^2
\]

Distributive Property

Step 2  Then write a function \( g \) that represents the translation of \( h \).

\[
g(x) = h(x) + 3
\]

Add 3 to the output.

\[
= 2x^4 - 4x^2 + 3
\]

Substitute \( 2x^4 - 4x^2 \) for \( h(x) \).

\[\text{The transformed function is } g(x) = 2x^4 - 4x^2 + 3.\]

EXAMPLE 5  Modeling with Mathematics

The function \( V(x) = \frac{1}{3}x^3 - x^2 \) represents the volume (in cubic feet) of the square pyramid shown. The function \( W(x) = V(3x) \) represents the volume (in cubic feet) when \( x \) is measured in yards. Write a rule for \( W \). Find and interpret \( W(10) \).

SOLUTION

1. Understand the Problem  You are given a function \( V \) whose inputs are in feet and whose outputs are in cubic feet. You are given another function \( W \) whose inputs are in yards and whose outputs are in cubic feet. The horizontal shrink shown by \( W(x) = V(3x) \) makes sense because there are 3 feet in 1 yard. You are asked to write a rule for \( W \) and interpret the output for a given input.

2. Make a Plan  Write the transformed function \( W(x) \) and then find \( W(10) \).

3. Solve the Problem  \( W(x) = V(3x) \)

\[
= \frac{1}{3}(3x)^3 - (3x)^2
\]

Replace \( x \) with \( 3x \) in \( V(x) \).

\[
= 9x^3 - 9x^2
\]

Simplify.

Next, find \( W(10) \).

\[
W(10) = 9(10)^3 - 9(10)^2 = 9000 - 900 = 8100
\]

\[\text{When } x \text{ is 10 yards, the volume of the pyramid is 8100 cubic feet.}\]

4. Look Back  Because \( W(10) = V(30) \), you can check that your solution is correct by verifying that \( V(30) = 8100 \).

\[
V(30) = \frac{1}{3}(30)^3 - (30)^2 = 9000 - 900 = 8100
\]

\[\checkmark\]

Monitoring Progress  Help in English and Spanish at BigIdeasMath.com

3. Let \( f(x) = x^5 - 4x + 6 \) and \( g(x) = -f(x) \). Write a rule for \( g \) and then graph each function. Describe the graph of \( g \) as a transformation of the graph of \( f \).

4. Let the graph of \( g \) be a horizontal stretch by a factor of 2, followed by a translation 3 units to the right of the graph of \( f(x) = 8x^3 + 3 \). Write a rule for \( g \).

5. WHAT IF? In Example 5, the height of the pyramid is \( 6x \), and the volume (in cubic feet) is represented by \( V(x) = 2x^3 \). Write a rule for \( W \). Find and interpret \( W(7) \).
3.7 Exercises

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The graph of \( f(x) = (x + 2)^3 \) is a __________ translation of the graph of \( f(x) = x^3 \).

2. **VOCABULARY** Describe how the vertex form of quadratic functions is similar to the form \( f(x) = a(x - h)^2 + k \) for cubic functions.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, describe the transformation of \( f \) represented by \( g \). Then graph each function. (See Example 1.)

3. \( f(x) = x^4 \), \( g(x) = x^4 + 3 \)
4. \( f(x) = x^4 \), \( g(x) = (x - 5)^4 \)
5. \( f(x) = x^5 \), \( g(x) = (x - 2)^5 - 1 \)
6. \( f(x) = x^6 \), \( g(x) = (x + 1)^6 - 4 \)

**ANALYZING RELATIONSHIPS** In Exercises 7–10, match the function with the correct transformation of the graph of \( f \). Explain your reasoning.

7. \( y = f(x - 2) \)
8. \( y = f(x + 2) + 2 \)
9. \( y = f(x - 2) + 2 \)
10. \( y = f(x) - 2 \)

A. ![Graph A]
B. ![Graph B]
C. ![Graph C]
D. ![Graph D]

In Exercises 11–16, describe the transformation of \( f \) represented by \( g \). Then graph each function. (See Example 2.)

11. \( f(x) = x^4 \), \( g(x) = -2x^4 \)
12. \( f(x) = x^6 \), \( g(x) = -3x^6 \)
13. \( f(x) = x^3 \), \( g(x) = 5x^3 + 1 \)
14. \( f(x) = x^4 \), \( g(x) = \frac{1}{2}x^4 + 1 \)
15. \( f(x) = x^5 \), \( g(x) = \frac{3}{4}(x + 4)^5 \)
16. \( f(x) = x^4 \), \( g(x) = (2x)^4 - 3 \)

In Exercises 17–20, write a rule for \( g \) and then graph each function. Describe the graph of \( g \) as a transformation of the graph of \( f \). (See Example 3.)

17. \( f(x) = x^4 + 1 \), \( g(x) = f(x + 2) \)
18. \( f(x) = x^5 - 2x + 3 \), \( g(x) = 3f(x) \)
19. \( f(x) = 2x^3 - 2x^2 + 6 \), \( g(x) = -\frac{1}{2}f(x) \)
20. \( f(x) = x^4 + x^3 - 1 \), \( g(x) = f(-x) - 5 \)

21. **ERROR ANALYSIS** Describe and correct the error in graphing the function \( g(x) = (x + 2)^3 - 6 \).

![Error Image]
22. ERROR ANALYSIS Describe and correct the error in describing the transformation of the graph of \( f(x) = x^3 \) represented by the graph of \( g(x) = (3x)^3 - 4 \).

The graph of \( g \) is a horizontal shrink by a factor of 3, followed by a translation 4 units down of the graph of \( f \).

In Exercises 23–26, write a rule for \( g \) that represents the indicated transformations of the graph of \( f \).

(See Example 4.)

23. \( f(x) = x^3 - 6; \) translation 3 units left, followed by a reflection in the \( y \)-axis

24. \( f(x) = x^4 + 2x + 6; \) vertical stretch by a factor of 2, followed by a translation 4 units right

25. \( f(x) = x^3 + 2x^2 - 9; \) horizontal shrink by a factor of \( \frac{1}{3} \) and a translation 2 units up, followed by a reflection in the \( x \)-axis

26. \( f(x) = 2x^3 - x^3 + x^2 + 4; \) reflection in the \( y \)-axis and a vertical stretch by a factor of 3, followed by a translation 1 unit down

27. MODELING WITH MATHEMATICS The volume \( V \) (in cubic feet) of the pyramid is given by \( V(x) = x^3 - 4x \). The function \( W(x) = V(3x) \) gives the volume (in cubic feet) of the pyramid when \( x \) is measured in yards. Write a rule for \( W \). Find and interpret \( W(5) \). (See Example 5.)

28. MAKING AN ARGUMENT The volume of a cube with side length \( x \) is given by \( V(x) = x^3 \). Your friend claims that when you divide the volume in half, the volume decreases by a greater amount than when you divide each side length in half. Is your friend correct? Justify your answer.

29. OPEN-ENDED Describe two transformations of the graph of \( f(x) = x^3 \) where the order in which the transformations are performed is important. Then describe two transformations where the order is not important. Explain your reasoning.

30. THOUGHT PROVOKING Write and graph a transformation of the graph of \( f(x) = x^3 - 3x^2 + 2x - 4 \) that results in a graph with a \( y \)-intercept of \(-2\).

31. PROBLEM SOLVING A portion of the path that a hummingbird flies while feeding can be modeled by the function \( f(x) = -\frac{1}{2}x(x - 4)^2(x - 7) \), \( 0 \leq x \leq 7 \) where \( x \) is the horizontal distance (in meters) and \( f(x) \) is the height (in meters). The hummingbird feeds each time it is at ground level.

a. At what distances does the hummingbird feed?

b. A second hummingbird feeds 2 meters farther away than the first hummingbird and flies twice as high. Write a function to model the path of the second hummingbird.

32. HOW DO YOU SEE IT? Determine the real zeros of each function. Then describe the transformation of the graph of \( f \) that results in the graph of \( g \).

33. MATHEMATICAL CONNECTIONS Write a function \( V \) for the volume (in cubic yards) of the right circular cone shown. Then write a function \( W \) that gives the volume (in cubic yards) of the cone when \( x \) is measured in feet. Find and interpret \( W(3) \).

Maintaining Mathematical Proficiency

Find the minimum value or maximum value of the function. Describe the domain and range of the function, and where the function is increasing and decreasing. (Section 2.5)

34. \( h(x) = (x + 5)^2 - 7 \)
35. \( f(x) = 4 - x^2 \)
36. \( f(x) = 3(x - 10)(x + 4) \)
37. \( g(x) = -(x + 2)(x + 8) \)
38. \( h(x) = \frac{1}{2}(x - 1)^2 - 3 \)
39. \( f(x) = -2x^2 + 4x - 1 \)
3.8 Analyzing Graphs of Polynomial Functions

Essential Question
How many turning points can the graph of a polynomial function have?
A turning point of the graph of a polynomial function is a point on the graph at which the function changes from
• increasing to decreasing, or
• decreasing to increasing.

EXPLORATION 1 Approximating Turning Points

Work with a partner. Match each polynomial function with its graph. Explain your reasoning. Then use a graphing calculator to approximate the coordinates of the turning points of the graph of the function. Round your answers to the nearest hundredth.

a. \( f(x) = 2x^2 + 3x - 4 \)
b. \( f(x) = x^2 + 3x + 2 \)
c. \( f(x) = x^3 - 2x^2 - x + 1 \)
d. \( f(x) = -x^3 + 5x - 2 \)
e. \( f(x) = x^4 - 3x^2 + 2x - 1 \)
f. \( f(x) = -2x^5 - x^2 + 5x + 3 \)

Communicate Your Answer

2. How many turning points can the graph of a polynomial function have?
3. Is it possible to sketch the graph of a cubic polynomial function that has no turning points? Justify your answer.

ATTENDING TO PRECISION
To be proficient in math, you need to express numerical answers with a degree of precision appropriate for the problem context.
3.8 Lesson

What You Will Learn

- Use x-intercepts to graph polynomial functions.
- Use the Location Principle to identify zeros of polynomial functions.
- Find turning points and identify local maximums and local minimums of graphs of polynomial functions.
- Identify even and odd functions.

Graphing Polynomial Functions

In this chapter, you have learned that zeros, factors, solutions, and x-intercepts are closely related concepts. Here is a summary of these relationships.

<table>
<thead>
<tr>
<th>Local maximum</th>
<th>p. 170</th>
</tr>
</thead>
<tbody>
<tr>
<td>Local minimum</td>
<td>p. 170</td>
</tr>
<tr>
<td>Even function</td>
<td>p. 171</td>
</tr>
<tr>
<td>Odd function</td>
<td>p. 171</td>
</tr>
</tbody>
</table>

Previous

- End behavior
- Increasing
decreasing
- Symmetric about the y-axis

Core Vocabulary

- Zeros, Factors, Solutions, and Intercepts

Let \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) be a polynomial function. The following statements are equivalent.

| Zero: \( k \) is a zero of the polynomial function \( f \). |
| Factor: \( x - k \) is a factor of the polynomial \( f(x) \). |
| Solution: \( k \) is a solution (or root) of the polynomial equation \( f(x) = 0 \). |
| x-Intercept: If \( k \) is a real number, then \( k \) is an x-intercept of the graph of the polynomial function \( f \). The graph of \( f \) passes through \((k, 0)\). |

Core Concept

Example 1

Using x-Intercepts to Graph a Polynomial Function

Graph the function

\[ f(x) = \frac{1}{6}(x + 3)(x - 2)^2. \]

**SOLUTION**

**Step 1** Plot the x-intercepts. Because \(-3\) and \(2\) are zeros of \( f \), plot \((-3, 0)\) and \((2, 0)\).

**Step 2** Plot points between and beyond the x-intercepts.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>(\frac{8}{3})</td>
<td>(3)</td>
<td>(2)</td>
<td>(\frac{2}{3})</td>
<td>(1)</td>
</tr>
</tbody>
</table>

**Step 3** Determine end behavior. Because \( f(x) \) has three factors of the form \( x - k \) and a constant factor of \( \frac{1}{6}, f \) is a cubic function with a positive leading coefficient. So, \( f(x) \rightarrow -\infty \) as \( x \rightarrow -\infty \) and \( f(x) \rightarrow +\infty \) as \( x \rightarrow +\infty \).

**Step 4** Draw the graph so that it passes through the plotted points and has the appropriate end behavior.

Monitoring Progress

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Graph the function.

1. \( f(x) = \frac{1}{2}(x + 1)(x - 4)^2 \)
2. \( f(x) = \frac{1}{4}(x + 2)(x - 1)(x - 3) \)

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The Location Principle

You can use the Location Principle to help you find real zeros of polynomial functions.

Core Concept

The Location Principle

If \( f \) is a polynomial function, and \( a \) and \( b \) are two real numbers such that \( f(a) < 0 \) and \( f(b) > 0 \), then \( f \) has at least one real zero between \( a \) and \( b \).

To use this principle to locate real zeros of a polynomial function, find a value \( a \) at which the polynomial function is negative and another value \( b \) at which the function is positive. You can conclude that the function has at least one real zero between \( a \) and \( b \).

EXAMPLE 2 Locating Real Zeros of a Polynomial Function

Find all real zeros of

\[
f(x) = 6x^3 + 5x^2 - 17x - 6.
\]

SOLUTION

Step 1 Use a graphing calculator to make a table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-6</td>
</tr>
<tr>
<td>1</td>
<td>-12</td>
</tr>
<tr>
<td>2</td>
<td>28</td>
</tr>
<tr>
<td>3</td>
<td>150</td>
</tr>
<tr>
<td>4</td>
<td>590</td>
</tr>
<tr>
<td>5</td>
<td>784</td>
</tr>
<tr>
<td>6</td>
<td>1368</td>
</tr>
</tbody>
</table>

Step 2 Use the Location Principle. From the table shown, you can see that \( f(1) < 0 \) and \( f(2) > 0 \). So, by the Location Principle, \( f \) has a zero between 1 and 2. Because \( f \) is a polynomial function of degree 3, it has three zeros. The only possible rational zero between 1 and 2 is \( \frac{3}{2} \). Using synthetic division, you can confirm that \( \frac{3}{2} \) is a zero.

Step 3 Write \( f(x) \) in factored form. Dividing \( f(x) \) by its known factor \( x - \frac{3}{2} \) gives a quotient of \( 6x^2 + 14x + 4 \). So, you can factor \( f(x) \) as

\[
f(x) = \left(x - \frac{3}{2}\right)(6x^2 + 14x + 4)
\]

\[
= 2\left(x - \frac{3}{2}\right)(3x^2 + 7x + 2)
\]

\[
= 2\left(x - \frac{3}{2}\right)(3x + 1)(x + 2).
\]

From the factorization, there are three zeros. The zeros of \( f \) are \( \frac{3}{2}, \frac{1}{3} \), and -2.

Check this by graphing \( f \).

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3. Find all real zeros of \( f(x) = 18x^3 + 21x^2 - 13x - 6 \).
Another important characteristic of graphs of polynomial functions is that they have **turning points** corresponding to local maximum and minimum values.

- The y-coordinate of a turning point is a **local maximum** of the function when the point is higher than all nearby points.
- The y-coordinate of a turning point is a **local minimum** of the function when the point is lower than all nearby points.

The turning points of a graph help determine the intervals for which a function is increasing or decreasing. You can write these intervals using interval notation.

**Core Concept**

**Turning Points of Polynomial Functions**

1. The graph of every polynomial function of degree $n$ has *at most* $n - 1$ turning points.

2. If a polynomial function has $n$ distinct real zeros, then its graph has *exactly* $n - 1$ turning points.

**Example 3** Finding Turning Points

Graph each function. Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which each function is increasing or decreasing.

**SOLUTION**

a. $f(x) = x^3 - 3x^2 + 6$

b. $g(x) = x^4 - 6x^3 + 3x^2 + 10x - 3$

**Monitoring Progress**

4. Graph $f(x) = 0.5x^3 + x^2 - x + 2$. Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which the function is increasing or decreasing.
Even and Odd Functions

A function $f$ is an **even function** when $f(-x) = f(x)$ for all $x$ in its domain. The graph of an even function is **symmetric about the y-axis**.

A function $f$ is an **odd function** when $f(-x) = -f(x)$ for all $x$ in its domain. The graph of an odd function is **symmetric about the origin**. One way to recognize a graph that is symmetric about the origin is that it looks the same after a $180^\circ$ rotation about the origin.

---

**EXAMPLE 4** Identifying Even and Odd Functions

Determine whether each function is *even, odd, or neither*.

**a.** $f(x) = x^3 - 7x$  
**b.** $g(x) = x^4 + x^2 - 1$  
**c.** $h(x) = x^3 + 2$

**SOLUTION**

**a.** Replace $x$ with $-x$ in the equation for $f$, and then simplify.

$$f(-x) = (-x)^3 - 7(-x) = -x^3 + 7x = -(x^3 - 7x) = -f(x)$$

Because $f(-x) = -f(x)$, the function is odd.

**b.** Replace $x$ with $-x$ in the equation for $g$, and then simplify.

$$g(-x) = (-x)^4 + (-x)^2 - 1 = x^4 + x^2 - 1 = g(x)$$

Because $g(-x) = g(x)$, the function is even.

**c.** Replacing $x$ with $-x$ in the equation for $h$ produces

$$h(-x) = (-x)^3 + 2 = -x^3 + 2.$$ 

Because $h(x) = x^3 + 2$ and $-h(x) = -x^3 - 2$, you can conclude that $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$. So, the function is neither even nor odd.

---

**Monitoring Progress**

Determine whether the function is *even, odd, or neither*.

5. $f(x) = -x^2 + 5$  
6. $f(x) = x^4 - 5x^3$  
7. $f(x) = 2x^5$
Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** A local maximum or local minimum of a polynomial function occurs at a __________ point of the graph of the function.

2. **WRITING** Explain what a local maximum of a function is and how it may be different from the maximum value of the function.

**Monitoring Progress and Modeling with Mathematics**

**ANALYZING RELATIONSHIPS** In Exercises 3–6, match the function with its graph.

3. \( f(x) = (x - 1)(x - 2)(x + 2) \)

4. \( h(x) = (x + 2)^2(x + 1) \)

5. \( g(x) = (x + 1)(x - 1)(x + 2) \)

6. \( f(x) = (x - 1)^2(x + 2) \)

**ERROR ANALYSIS** In Exercises 15 and 16, describe and correct the error in using factors to graph \( f \).

15. \( f(x) = (x + 2)(x - 1)^2 \)

**In Exercises 17–22, find all real zeros of the function. (See Example 2.)**

17. \( f(x) = x^3 - 4x^2 - x + 4 \)

18. \( f(x) = x^3 - 3x^2 - 4x + 12 \)

19. \( h(x) = 2x^3 + 7x^2 - 5x - 4 \)

20. \( h(x) = 4x^3 - 2x^2 - 24x - 18 \)

21. \( g(x) = 4x^3 + x^2 - 51x + 36 \)

22. \( f(x) = 2x^3 - 3x^2 - 32x - 15 \)
In Exercises 23–30, graph the function. Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which the function is increasing or decreasing. (See Example 3.)

23. \( g(x) = 2x^3 + 8x^2 - 3 \)
24. \( g(x) = -x^4 + 3x \)
25. \( h(x) = x^3 - 3x^2 + x \)
26. \( f(x) = x^5 - 4x^3 + x^2 + 2 \)
27. \( f(x) = 0.5x^3 - 2x + 2.5 \)
28. \( f(x) = 0.7x^4 - 3x^3 + 5x \)
29. \( h(x) = x^5 + 2x^2 - 17x - 4 \)
30. \( g(x) = x^4 - 5x^3 + 2x^2 + x - 3 \)

In Exercises 31–36, estimate the coordinates of each turning point. State whether each corresponds to a local maximum or a local minimum. Then estimate the real zeros and find the least possible degree of the function.

31.

32.

33.

34.

35.

36.

OPEN-ENDED In Exercises 37 and 38, sketch a graph of a polynomial function \( f \) having the given characteristics.

37. \( f \) has x-intercepts at \( x = -4, x = 0, \) and \( x = 2. \)
   - \( f \) has a local maximum value when \( x = 1. \)
   - \( f \) has a local minimum value when \( x = -2. \)

38. \( f \) has x-intercepts at \( x = -3, x = 1, \) and \( x = 5. \)
   - \( f \) has a local maximum value when \( x = 1. \)
   - \( f \) has a local minimum value when \( x = -2 \) and when \( x = 4. \)

In Exercises 39–46, determine whether the function is even, odd, or neither. (See Example 4.)

39. \( h(x) = 4x^7 \)
40. \( g(x) = -2x^6 + x^3 \)
41. \( f(x) = x^4 + 3x^2 - 2 \)
42. \( f(x) = x^5 + 3x^3 - 2 \)
43. \( g(x) = x^2 + 5x + 1 \)
44. \( f(x) = -x^3 + 2x - 9 \)
45. \( f(x) = x^4 - 12x^2 \)
46. \( h(x) = x^5 + 3x^4 \)

47. USING TOOLS When a swimmer does the breaststroke, the function
   \[
   S = -241t^7 + 1060t^6 - 1870t^5 + 1650t^4 - 737t^3 + 144t^2 - 2.43t
   \]
   models the speed \( S \) (in meters per second) of the swimmer during one complete stroke, where \( t \) is the number of seconds since the start of the stroke and \( 0 \leq t \leq 1.22 \). Use a graphing calculator to graph the function. At what time during the stroke is the swimmer traveling the fastest?

48. USING TOOLS During a recent period of time, the number \( S \) (in thousands) of students enrolled in public schools in a certain country can be modeled by
   \[
   S = 1.64x^3 - 102x^2 + 1710x + 36,300, \text{ where } x \text{ is time (in years). Use a graphing calculator to graph the function for the interval } 0 \leq x \leq 41. \text{ Then describe how the public school enrollment changes over this period of time.}
   \]

49. WRITING Why is the adjective local, used to describe the maximums and minimums of cubic functions, sometimes not required for quadratic functions?
50. **HOW DO YOU SEE IT?** The graph of a polynomial function is shown.

![Graph of a polynomial function](image)

a. Find the zeros, local maximum, and local minimum values of the function.
b. Compare the x-intercepts of the graphs of \( y = f(x) \) and \( y = -f(x) \).
c. Compare the maximum and minimum values of the functions \( y = f(x) \) and \( y = -f(x) \).

51. **MAKING AN ARGUMENT** Your friend claims that the product of two odd functions is an odd function. Is your friend correct? Explain your reasoning.

52. **MODELING WITH MATHEMATICS** You are making a rectangular box out of a 16-inch-by-20-inch piece of cardboard. The box will be formed by making the cuts shown in the diagram and folding up the sides. You want the box to have the greatest volume possible.

![Diagram of a box being made](image)

a. How long should you make the cuts?
b. What is the maximum volume?
c. What are the dimensions of the finished box?

53. **PROBLEM SOLVING** Quonset huts are temporary, all-purpose structures shaped like half-cylinders. You have 1100 square feet of material to build a quonset hut.

a. The surface area \( S \) of a quonset hut is given by \( S = \pi r^2 + \pi \ell \). Substitute 1100 for \( S \) and then write an expression for \( \ell \) in terms of \( r \).
b. The volume \( V \) of a quonset hut is given by \( V = \frac{1}{2} \pi r^2 \ell \). Write an equation that gives \( V \) as a function in terms of \( r \) only.
c. Find the value of \( r \) that maximizes the volume of the hut.

![Image of a quonset hut](image)

54. **THOUGHT PROVOKING** Write and graph a polynomial function that has one real zero in each of the intervals \(-2 < x < -1\), \(0 < x < 1\), and \(4 < x < 5\). Is there a maximum degree that such a polynomial function can have? Justify your answer.

55. **MATHEMATICAL CONNECTIONS** A cylinder is inscribed in a sphere of radius 8 inches. Write an equation for the volume of the cylinder as a function of \( h \). Find the value of \( h \) that maximizes the volume of the inscribed cylinder. What is the maximum volume of the cylinder?

![Diagram of a cylinder inscribed in a sphere](image)

56. **Maintaining Mathematical Proficiency** Reviewing what you learned in previous grades and lessons.

State whether the table displays **linear data**, **quadratic data**, or **neither**. Explain. (Section 2.7)

<table>
<thead>
<tr>
<th>Months, ( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Savings (dollars), ( y )</td>
<td>100</td>
<td>150</td>
<td>200</td>
<td>250</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time (seconds), ( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (feet), ( y )</td>
<td>300</td>
<td>284</td>
<td>236</td>
<td>156</td>
</tr>
</tbody>
</table>
Section 3.9  Modeling with Polynomial Functions

Essential Question  How can you find a polynomial model for real-life data?

Exploration 1  Modeling Real-Life Data

Work with a partner. The distance a baseball travels after it is hit depends on the angle at which it was hit and the initial speed. The table shows the distances a baseball hit at an angle of 35° travels at various initial speeds.

<table>
<thead>
<tr>
<th>Initial speed, x (miles per hour)</th>
<th>80</th>
<th>85</th>
<th>90</th>
<th>95</th>
<th>100</th>
<th>105</th>
<th>110</th>
<th>115</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance, y (feet)</td>
<td>194</td>
<td>220</td>
<td>247</td>
<td>275</td>
<td>304</td>
<td>334</td>
<td>365</td>
<td>397</td>
</tr>
</tbody>
</table>

a. Recall that when data have equally-spaced x-values, you can analyze patterns in the differences of the y-values to determine what type of function can be used to model the data. If the first differences are constant, then the set of data fits a linear model. If the second differences are constant, then the set of data fits a quadratic model.

Find the first and second differences of the data. Are the data linear or quadratic? Explain your reasoning.

b. Use a graphing calculator to draw a scatter plot of the data. Do the data appear linear or quadratic? Use the regression feature of the graphing calculator to find a linear or quadratic model that best fits the data.

c. Use the model you found in part (b) to find the distance a baseball travels when it is hit at an angle of 35° and travels at an initial speed of 120 miles per hour.

d. According to the Baseball Almanac, “Any drive over 400 feet is noteworthy. A blow of 450 feet shows exceptional power, as the majority of major league players are unable to hit a ball that far. Anything in the 500-foot range is genuinely historic.” Estimate the initial speed of a baseball that travels a distance of 500 feet.

Communicate Your Answer

2. How can you find a polynomial model for real-life data?

3. How well does the model you found in Exploration 1(b) fit the data? Do you think the model is valid for any initial speed? Explain your reasoning.
### What You Will Learn

- Write polynomial functions for sets of points.
- Write polynomial functions using finite differences.
- Use technology to find models for data sets.

### Writing Polynomial Functions for a Set of Points

You know that two points determine a line and three points not on a line determine a parabola. In Example 1, you will see that four points not on a line or a parabola determine the graph of a cubic function.

**Writing a Cubic Function**

Write the cubic function whose graph is shown.

**SOLUTION**

**Step 1** Use the three $x$-intercepts to write the function in factored form.

$$f(x) = a(x + 4)(x - 1)(x - 3)$$

**Step 2** Find the value of $a$ by substituting the coordinates of the point $(0, -6)$.

$$-6 = a(0 + 4)(0 - 1)(0 - 3)$$

$$-6 = 12a$$

$$a = -\frac{1}{2}$$

The function is $f(x) = -\frac{1}{2}(x + 4)(x - 1)(x - 3)$.

### Monitoring Progress

Write a cubic function whose graph passes through the given points.

1. $(-4, 0), (0, 10), (2, 0), (5, 0)$
2. $(-1, 0), (0, -12), (2, 0), (3, 0)$

### Finite Differences

When the $x$-values in a data set are equally spaced, the differences of consecutive $y$-values are called finite differences. Recall from Section 2.7 that the first and second differences of $y = x^2$ are:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-3$</td>
<td>9</td>
</tr>
<tr>
<td>$-2$</td>
<td>4</td>
</tr>
<tr>
<td>$-1$</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
</tbody>
</table>

First differences: $-5, -3, -1, 3, 5$

Second differences: $2, 2, 2, 2, 2$

Notice that $y = x^2$ has degree two and that the second differences are constant and nonzero. This illustrates the first of the two properties of finite differences shown on the next page.
Properties of Finite Differences

1. If a polynomial function \( y = f(x) \) has degree \( n \), then the \( n \)th differences of function values for equally-spaced \( x \)-values are nonzero and constant.

2. Conversely, if the \( n \)th differences of equally-spaced data are nonzero and constant, then the data can be represented by a polynomial function of degree \( n \).

The second property of finite differences allows you to write a polynomial function that models a set of equally-spaced data.

**EXAMPLE 2** Writing a Function Using Finite Differences

Use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function.

**SOLUTION**

**Step 1** Write the function values. Find the first differences by subtracting consecutive values. Then find the second differences by subtracting consecutive first differences. Continue until you obtain differences that are nonzero and constant.

\[
\begin{array}{c|cccccccc}
 x & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
 f(x) & 1 & 4 & 10 & 20 & 35 & 56 & 84 \\
\end{array}
\]

Because the third differences are nonzero and constant, you can model the data **exactly** with a cubic function.

**Step 2** Enter the data into a graphing calculator and use cubic regression to obtain a polynomial function.

\[
y = ax^3 + bx^2 + cx + d
\]

\[
a = 0.1666666667
\]

\[
b = 0.5
\]

\[
c = 0.3333333333
\]

\[
d = 0
\]

**R^2 = 1**

3. Use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function.

\[
\begin{array}{c|cccccc}
 x & -3 & -2 & -1 & 0 & 1 & 2 \\
 f(x) & 6 & 15 & 22 & 21 & 6 & -29 \\
\end{array}
\]
Finding Models Using Technology

In Examples 1 and 2, you found a cubic model that exactly fits a set of data. In many real-life situations, you cannot find models to fit data exactly. Despite this limitation, you can still use technology to approximate the data with a polynomial model, as shown in the next example.

EXAMPLE 3 Real-Life Application

The table shows the total U.S. biomass energy consumptions \( y \) (in trillions of British thermal units, or Btus) in the year \( t \), where \( t = 1 \) corresponds to 2001. Find a polynomial model for the data. Use the model to estimate the total U.S. biomass energy consumption in 2013.

<table>
<thead>
<tr>
<th>( t )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>2622</td>
<td>2701</td>
<td>2807</td>
<td>3010</td>
<td>3117</td>
<td>3267</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( t )</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>3493</td>
<td>3866</td>
<td>3951</td>
<td>4286</td>
<td>4421</td>
<td>4316</td>
</tr>
</tbody>
</table>

SOLUTION

Step 1 Enter the data into a graphing calculator and make a scatter plot. The data suggest a cubic model.

Step 2 Use the cubic regression feature. The polynomial model is

\[
y = -2.545t^3 + 51.95t^2 - 118.1t + 2732.
\]

Step 3 Check the model by graphing it and the data in the same viewing window.

Step 4 Use the trace feature to estimate the value of the model when \( t = 13 \).

The approximate total U.S. biomass energy consumption in 2013 was about 4385 trillion Btus.

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Use a graphing calculator to find a polynomial function that fits the data.

4. \[
\begin{array}{ccccccc}
\text{x} & 1 & 2 & 3 & 4 & 5 & 6 \\
y & 5 & 13 & 17 & 11 & 11 & 56
\end{array}
\]

5. \[
\begin{array}{cccccccc}
\text{x} & 0 & 2 & 4 & 6 & 8 & 10 \\
y & 8 & 0 & 15 & 69 & 98 & 87
\end{array}
\]
3.9 Exercises

Vocabulary and Core Concept Check

1. COMPLETE THE SENTENCE When the x-values in a set of data are equally spaced, the differences of consecutive y-values are called ________________.

2. WRITING Explain how you know when a set of data could be modeled by a cubic function.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, write a cubic function whose graph is shown. (See Example 1.)

3.  

![Graph 3](image)

4.  

![Graph 4](image)

5.  

![Graph 5](image)

6.  

![Graph 6](image)

In Exercises 7–12, use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function. (See Example 2.)

7.  

<table>
<thead>
<tr>
<th>x</th>
<th>-6</th>
<th>-3</th>
<th>0</th>
<th>3</th>
<th>6</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>-2</td>
<td>15</td>
<td>-4</td>
<td>49</td>
<td>282</td>
<td>803</td>
</tr>
</tbody>
</table>

8.  

<table>
<thead>
<tr>
<th>x</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>-14</td>
<td>-5</td>
<td>-2</td>
<td>7</td>
<td>34</td>
<td>91</td>
</tr>
</tbody>
</table>

9.  

(-4, -317), (-3, -37), (-2, 21), (-1, 7), (0, -1), (1, 3), (2, -47), (3, -289), (4, -933)

10.  

(-6, 744), (-4, 154), (-2, 4), (0, -6), (2, 16), (4, 154), (6, 684), (8, 2074), (10, 4984)

11.  

(-2, 968), (-1, 422), (0, 142), (1, 26), (2, -4), (3, -2), (4, 2), (5, 2), (6, 16)

12.  

(1, 0), (2, 6), (3, 2), (4, 6), (5, 12), (6, -10), (7, -114), (8, -378), (9, -904)

13. ERROR ANALYSIS Describe and correct the error in writing a cubic function whose graph passes through the given points.

\[ (-6, 0), (1, 0), (3, 0), (0, 54) \]

\[ 54 = a(0 - 6)(0 + 1)(0 + 3) \]

\[ a = -3 \]

\[ f(x) = -3(x - 6)(x + 1)(x + 3) \]

14. MODELING WITH MATHEMATICS The dot patterns show pentagonal numbers. The number of dots in the nth pentagonal number is given by \[ f(n) = \frac{1}{2}n(3n - 1). \] Show that this function has constant second-order differences.

15. OPEN-ENDED Write three different cubic functions that pass through the points (3, 0), (4, 0), and (2, 6). Justify your answers.

16. MODELING WITH MATHEMATICS The table shows the ages of cats and their corresponding ages in human years. Find a polynomial model for the data for the first 8 years of a cat’s life. Use the model to estimate the age (in human years) of a cat that is 3 years old. (See Example 3.)

<table>
<thead>
<tr>
<th>Age of cat, x</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Human years, y</td>
<td>15</td>
<td>24</td>
<td>32</td>
<td>40</td>
<td>44</td>
<td>48</td>
</tr>
</tbody>
</table>

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17. **MODELING WITH MATHEMATICS** The data in the table show the average speeds \( y \) (in miles per hour) of a pontoon boat for several different engine speeds \( x \) (in hundreds of revolutions per minute, or RPMs). Find a polynomial model for the data. Estimate the average speed of the pontoon boat when the engine speed is 2800 RPMs.

<table>
<thead>
<tr>
<th>( x )</th>
<th>10</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>45</th>
<th>55</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>4.5</td>
<td>8.9</td>
<td>13.8</td>
<td>18.9</td>
<td>29.9</td>
<td>37.7</td>
</tr>
</tbody>
</table>

18. **HOW DO YOU SEE IT?** The graph shows typical speeds \( y \) (in feet per second) of a space shuttle \( x \) seconds after it is launched.

![Space Launch Graph](image)

a. What type of polynomial function models the data? Explain.

b. Which \( n \)-th order finite difference should be constant for the function in part (a)? Explain.

19. **MATHEMATICAL CONNECTIONS** The table shows the number of diagonals for polygons with \( n \) sides. Find a polynomial function that fits the data. Determine the total number of diagonals in the decagon shown.

<table>
<thead>
<tr>
<th>Number of sides, ( n )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of diagonals, ( d )</td>
<td>0</td>
<td>2</td>
<td>5</td>
<td>9</td>
<td>14</td>
<td>20</td>
</tr>
</tbody>
</table>

20. **MAKING AN ARGUMENT** Your friend states that it is not possible to determine the degree of a function given the first-order differences. Is your friend correct? Explain your reasoning.

21. **WRITING** Explain why you cannot always use finite differences to find a model for real-life data sets.

22. **THOUGHT PROVOKING** \( A, B, \) and \( C \) are zeros of a cubic polynomial function. Choose values for \( A, B, \) and \( C \) such that the distance from \( A \) to \( B \) is less than or equal to the distance from \( A \) to \( C \). Then write the function using the \( A, B, \) and \( C \) values you chose.

23. **MULTIPLE REPRESENTATIONS** Order the polynomial functions according to their degree, from least to greatest.

A. \( f(x) = -3x + 2x^2 + 1 \)

B. \( g(x) = x^2 + 2x + 3 \)

C. \( h(x) = 2x^3 + x^2 + 3x + 4 \)

D. \( k(x) = x^3 + 2x^2 + 3x + 4 \)

24. **ABSTRACT REASONING** Substitute the expressions \( z, z + 1, z + 2, \ldots, z + 5 \) for \( x \) in the function \( f(x) = ax^3 + bx^2 + cx + d \) to generate six equally-spaced ordered pairs. Then show that the third-order differences are constant.

---

**Maintaining Mathematical Proficiency**

Solve the equation using square roots. *(Skills Review Handbook)*

25. \( x^2 - 6 = 30 \)

26. \( 5x^2 - 38 = 187 \)

27. \( 2(x - 3)^2 = 24 \)

28. \( \frac{4}{3}(x + 5)^2 = 4 \)

Solve the equation using the Quadratic Formula. *(Skills Review Handbook)*

29. \( 2x^2 + 3x = 5 \)

30. \( 2x^2 + \frac{1}{2} = 2x \)

31. \( 2x^2 + 3x = -3x^2 + 1 \)

32. \( 4x - 20 = x^2 \)
3.5–3.9 **What Did You Learn?**

### Core Vocabulary
- repeated solution, p. 146
- complex conjugates, p. 155
- local minimum, p. 170
- finite differences, p. 176
- even function, p. 171
- odd function, p. 171
- local maximum, p. 170

### Core Concepts

#### Section 3.5
- The Rational Root Theorem, p. 147
- The Irrational Conjugates Theorem, p. 149

#### Section 3.6
- The Fundamental Theorem of Algebra, p. 154
- The Complex Conjugates Theorem, p. 155
- Descartes’s Rule of Signs, p. 156

#### Section 3.7
- Transformations of Polynomial Functions, p. 162
- Writing Transformed Polynomial Functions, p. 163

#### Section 3.8
- Zeros, Factors, Solutions, and Intercepts, p. 168
- Turning Points of Polynomial Functions, p. 170
- Even and Odd Functions, p. 171
- The Location Principle, p. 169

#### Section 3.9
- Writing Polynomial Functions for Data Sets, p. 176
- Properties of Finite Differences, p. 177

### Mathematical Practices

1. Explain how understanding the Complex Conjugates Theorem allows you to construct your argument in Exercise 46 on page 159.

2. Describe how you use structure to accurately match each graph with its transformation in Exercises 7–10 on page 165.

### Performance Task:

**Quonset Huts**

Over 153,000 Quonset huts were procured by the United States Navy during the 1940s. The most common huts were 20 feet wide and 48 feet long. How many different sizes of Quonset huts can you design that have approximately the same volume as this model? How do the surface areas of your new huts compare to the original model?

To explore the answers to these questions and more, check out the Performance Task and Real-Life STEM video at [BigIdeasMath.com](http://BigIdeasMath.com).
3.1 Graphing Polynomial Functions (pp. 111–118)

Graph \( f(x) = x^3 + 3x^2 - 3x - 10 \).

To graph the function, make a table of values and plot the corresponding points. Connect the points with a smooth curve and check the end behavior.

\[
\begin{array}{ccccccc}
 x & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
 f(x) & -1 & 0 & -5 & -10 & -9 & 4 & 35 \\
\end{array}
\]

The degree is odd and the leading coefficient is positive. So, \( f(x) \to -\infty \) as \( x \to -\infty \) and \( f(x) \to +\infty \) as \( x \to +\infty \).

Decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

1. \( h(x) = -x^3 + 2x^2 - 15x^2 \)  
2. \( p(x) = x^3 - 5x^{0.5} + 13x^2 + 8 \)

Graph the polynomial function.

3. \( h(x) = x^2 + 6x^5 - 5 \)  
4. \( f(x) = 3x^4 - 5x^2 + 1 \)  
5. \( g(x) = -x^4 + x + 2 \)

3.2 Adding, Subtracting, and Multiplying Polynomials (pp. 119–128)

a. Multiply \((x - 2), (x - 1), \) and \( (x + 3) \) in a horizontal format.

\[
(x - 2)(x - 1)(x + 3) = (x^2 - 3x + 2)(x + 3) \\
= (x^2 - 3x + 2)x + (x^2 - 3x + 2)3 \\
= x^3 - 3x^2 + 2x + 3x^2 - 9x + 6 \\
= x^3 - 7x + 6
\]

b. Use Pascal’s Triangle to expand \((4x + 2)^4\).

The coefficients from the fourth row of Pascal’s Triangle are 1, 4, 6, 4, and 1.

\[
(4x + 2)^4 = 1(4x)^4 + 4(4x)^3(2) + 6(4x)^2(2)^2 + 4(4x)(2)^3 + 1(2)^4 \\
= 256x^4 + 512x^3 + 384x^2 + 128x + 16
\]

Find the sum or difference.

6. \( (4x^3 - 12x^2 - 5) - (-8x^2 + 4x + 3) \)  
7. \( (x^4 + 3x^3 - x^2 + 6) + (2x^4 - 3x + 9) \)  
8. \( (3x^2 + 9x + 13) - (x^2 - 2x + 12) \)

Find the product.

9. \( (2y^2 + 4y - 7)(y + 3) \)  
10. \( \left(2m + n\right)^3 \)  
11. \( (s + 2)(s + 4)(s - 3) \)

Use Pascal’s Triangle or the Binomial Theorem to expand the binomial.

12. \( (m + 4)^4 \)  
13. \( (3s + 2)^5 \)  
14. \( (z + 1)^6 \)
3.3 Dividing Polynomials (pp. 129–134)

Use synthetic division to evaluate \( f(x) = -2x^3 + 4x^2 + 8x + 10 \) when \( x = -3 \).

\[
\begin{array}{c|cccc}
-3 & -2 & 4 & 8 & 10 \\
& & 6 & -30 & 66 \\
-2 & 10 & -22 & 76 \\
\end{array}
\]

The remainder is 76. So, you can conclude from the Remainder Theorem that \( f(-3) = 76 \). You can check this by substituting \( x = -3 \) in the original function.

Check

\[
f(-3) = -2(-3)^3 + 4(-3)^2 + 8(-3) + 10
\]
\[
= 54 + 36 - 24 + 10
\]
\[
= 76 \checkmark
\]

Divide using polynomial long division or synthetic division.

15. \( (x^3 + x^2 + 3x - 4) \div (x^2 + 2x + 1) \)
16. \( (x^4 + 3x^3 - 4x^2 + 5x + 3) \div (x^2 + x + 4) \)
17. \( (x^4 - x^3 - 7) \div (x + 4) \)
18. Use synthetic division to evaluate \( g(x) = 4x^3 + 2x^2 - 4 \) when \( x = 5 \).

3.4 Factoring Polynomials (pp. 135–142)

a. Factor \( x^4 + 8x \) completely.

\[
x^4 + 8x = x(x^3 + 8)
\]
\[
= x(x^3 + 2^3)
\]
\[
= x(x + 2)(x^2 - 2x + 4)
\]

Factor common monomial.

Write \( x^3 + 8 \) as \( a^3 + b^3 \).

Sum of Two Cubes Pattern

b. Determine whether \( x + 4 \) is a factor of \( f(x) = x^5 + 4x^4 + 2x + 8 \).

Find \( f(-4) \) by synthetic division.

\[
\begin{array}{c|cccccc}
-4 & 1 & 4 & 0 & 0 & 2 & 8 \\
& & -4 & 0 & 0 & 0 & -8 \\
& 1 & 0 & 0 & 0 & 2 & 0 \\
\end{array}
\]

Because \( f(-4) = 0 \), the binomial \( x + 4 \) is a factor of \( f(x) = x^5 + 4x^4 + 2x + 8 \).

Factor the polynomial completely.

19. \( 64x^3 - 8 \)
20. \( 2x^5 - 12x^3 + 10z \)
21. \( 2a^3 - 7a^2 - 8a + 28 \)
22. Show that \( x + 2 \) is a factor of \( f(x) = x^4 + 2x^3 - 27x - 54 \). Then factor \( f(x) \) completely.
3.5 Solving Polynomial Equations (pp. 145–152)

a. Find all real solutions of \( x^3 + x^2 - 8x - 12 = 0 \).

**Step 1** List the possible rational solutions. The leading coefficient of the polynomial \( f(x) = x^3 + x^2 - 8x - 12 \) is 1, and the constant term is \(-12\). So, the possible rational solutions of \( f(x) = 0 \) are

\[
x = \pm \frac{1}{1}, \pm \frac{2}{1}, \pm \frac{3}{1}, \pm \frac{4}{1}, \pm \frac{6}{1}, \pm \frac{12}{1}.
\]

**Step 2** Test possible solutions using synthetic division until a solution is found.

\[
\begin{array}{c|ccc}
2 & 1 & 1 & -8 & -12 \\
 & & 2 & 6 & -4 \\
\hline
 & 1 & 3 & -2 & -16
\end{array}
\quad
\begin{array}{c|ccc}
-2 & 1 & 1 & -8 & -12 \\
 & & -2 & 2 & 12 \\
\hline
 & 1 & -1 & -6 & 0
\end{array}
\]

\( f(2) \neq 0 \), so \( x - 2 \) is not a factor of \( f(x) \).

\( f(-2) = 0 \), so \( x + 2 \) is a factor of \( f(x) \).

**Step 3** Factor completely using the result of synthetic division.

\[
(x + 2)(x^2 - x - 6) = 0
\]

Write as a product of factors.

\[
(x + 2)(x + 2)(x - 3) = 0
\]

Factor the trinomial.

So, the solutions are \( x = -2 \) and \( x = 3 \).

b. Write a polynomial function \( f \) of least degree that has rational coefficients, a leading coefficient of 1, and the zeros \(-4\) and \(1 + \sqrt{2}\).

By the Irrational Conjugates Theorem, \( 1 - \sqrt{2} \) must also be a zero of \( f \).

\[
f(x) = (x + 4)[x - (1 + \sqrt{2})][x - (1 - \sqrt{2})]
\]

Write \( f(x) \) in factored form.

\[
= (x + 4)[(x - 1) - \sqrt{2}][(x - 1) + \sqrt{2}]
\]

Regroup terms.

\[
= (x + 4)(x - 1)^2 - 2
\]

Multiply.

\[
= (x + 4)(x^2 - 2x + 1) - 2
\]

Expand binomial.

\[
= (x + 4)(x^2 - 2x - 1)
\]

Simplify.

\[
= x^3 - 2x^2 - x + 4x^2 - 8x - 4
\]

Multiply.

\[
= x^3 + 2x^2 - 9x - 4
\]

Combine like terms.

Find all real solutions of the equation.

23. \( x^3 + 3x^2 - 10x - 24 = 0 \)  
24. \( x^3 + 5x^2 - 2x - 24 = 0 \)

Write a polynomial function \( f \) of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros.

25. \(1, 2 - \sqrt{3}\)  
26. \(2, 3, \sqrt{5}\)  
27. \(-2, 5, 3 + \sqrt{6}\)

28. You use 240 cubic inches of clay to make a sculpture shaped as a rectangular prism. The width is 4 inches less than the length and the height is 2 inches more than three times the length. What are the dimensions of the sculpture? Justify your answer.
### 3.6 The Fundamental Theorem of Algebra  
(pp. 153–160)

Find all zeros of \( f(x) = x^4 + 2x^3 + 6x^2 + 18x - 27 \).

**Step 1** Find the rational zeros of \( f \). Because \( f \) is a polynomial function of degree 4, it has four zeros. The possible rational zeros are \( \pm 1, \pm 3, \pm 9, \) and \( \pm 27 \). Using synthetic division, you can determine that 1 is a zero and \(-3\) is also a zero.

**Step 2** Write \( f(x) \) in factored form. Dividing \( f(x) \) by its known factors \( x - 1 \) and \( x + 3 \) gives a quotient of \( x^2 + 9 \). So, \( f(x) = (x - 1)(x + 3)(x^2 + 9) \).

**Step 3** Find the complex zeros of \( f \). Solving \( x^2 + 9 = 0 \), you get \( x = \pm 3i \). This means \( x^2 + 9 = (x + 3i)(x - 3i) \). From the factorization, there are four zeros. The zeros of \( f \) are 1, \(-3\), \(-3i\), and \( 3i \).

Write a polynomial function \( f \) of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros.

29. \( 3, 1 + 2i \)  
30. \( -1, 2, 4i \)  
31. \( -5, -4, 1 - \sqrt{3} \)

Determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for the function.

32. \( f(x) = x^4 - 10x + 8 \)  
33. \( f(x) = -6x^4 - x^3 + 3x^2 + 2x + 18 \)

### 3.7 Transformations of Polynomial Functions  
(pp. 161–166)

Describe the transformation of \( f(x) = x^3 \) represented by \( g(x) = (x - 6)^3 - 2 \). Then graph each function.

Notice that the function is of the form \( g(x) = (x - h)^3 + k \). Rewrite the function to identify \( h \) and \( k \).

\[
g(x) = (x - 6)^3 + (-2)
\]

Because \( h = 6 \) and \( k = -2 \), the graph of \( g \) is a translation 6 units right and 2 units down of the graph of \( f \).

Describe the transformation of \( f \) represented by \( g \). Then graph each function.

34. \( f(x) = x^3, g(x) = (-x)^3 + 2 \)  
35. \( f(x) = x^4, g(x) = -(x + 9)^4 \)

Write a rule for \( g \).

36. Let the graph of \( g \) be a horizontal stretch by a factor of 4, followed by a translation 3 units right and 5 units down of the graph of \( f(x) = x^3 + 3x \).

37. Let the graph of \( g \) be a translation 5 units up, followed by a reflection in the \( y \)-axis of the graph of \( f(x) = x^3 - 2x^3 - 12 \).
### 3.8 Analyzing Graphs of Polynomial Functions (pp. 167–174)

Graph the function \( f(x) = x(x + 2)(x - 2) \). Then estimate the points where the local maximums and local minimums occur.

**Step 1** Plot the \( x \)-intercepts. Because \(-2, 0, \) and \(2\) are zeros of \( f \), plot \((-2, 0), (0, 0), \) and \( (2, 0) \).

**Step 2** Plot points between and beyond the \( x \)-intercepts.

- **Step 3** Determine end behavior. Because \( f(x) \) has three factors of the form \( x - k \) and a constant factor of \( 1 \), \( f \) is a cubic function with a positive leading coefficient. So \( f(x) \rightarrow -\infty \) as \( x \rightarrow -\infty \) and \( f(x) \rightarrow +\infty \) as \( x \rightarrow +\infty \).

**Step 4** Draw the graph so it passes through the plotted points and has the appropriate end behavior.

The function has a local maximum at \((-1.15, 3.08)\) and a local minimum at \((1.15, -3.08)\).

Graph the function. Identify the \( x \)-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which the function is increasing or decreasing.

38. \( f(x) = -2x^3 - 3x^2 - 1 \)

39. \( f(x) = x^4 + 3x^3 - x^2 - 8x + 2 \)

Determine whether the function is even, odd, or neither.

40. \( f(x) = 2x^3 + 3x \)

41. \( g(x) = 3x^2 - 7 \)

42. \( h(x) = x^5 + 3x^5 \)

### 3.9 Modeling with Polynomial Functions (pp. 175–180)

Write the cubic function whose graph is shown.

**Step 1** Use the three \( x \)-intercepts to write the function in factored form.

\[
f(x) = a(x + 3)(x + 1)(x - 2)
\]

**Step 2** Find the value of \( a \) by substituting the coordinates of the point \((0, -12)\).

\[
-12 = a(0 + 3)(0 + 1)(0 - 2)
\]

\[
-12 = -6a
\]

\[
a = 2
\]

The function is \( f(x) = 2(x + 3)(x + 1)(x - 2) \).

43. Write a cubic function whose graph passes through the points \((-4, 0), (4, 0), (0, 6), \) and \((2, 0)\).

44. Use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
<th>( 6 )</th>
<th>( 7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>(-11)</td>
<td>(-24)</td>
<td>(-27)</td>
<td>(-8)</td>
<td>(45)</td>
<td>(144)</td>
<td>(301)</td>
</tr>
</tbody>
</table>
Chapter Test

Write a polynomial function \( f \) of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros.

1. \( 3, 1 - \sqrt{2} \)  
2. \( -2, 4, 3i \)

Find the product or quotient.

3. \((x^6 - 4)(x^2 - 7x + 5)\)  
4. \((3x^4 - 2x^3 - x - 1) \div (x^2 - 2x + 1)\)  
5. \((2x^3 - 3x^2 + 5x - 1) \div (x + 2)\)  
6. \((2x + 3)^3\)

7. The graphs of \( f(x) = x^4 \) and \( g(x) = (x - 3)^4 \) are shown.
   a. How many zeros does each function have? Explain.
   b. Describe the transformation of \( f \) represented by \( g \).
   c. Determine the intervals for which the function \( g \) is increasing or decreasing.

8. The volume \( V \) (in cubic feet) of an aquarium is modeled by the polynomial function \( V(x) = x^3 + 2x^2 - 13x + 10 \), where \( x \) is the length of the tank.
   a. Explain how you know \( x = 4 \) is not a possible rational zero.
   b. Show that \( x - 1 \) is a factor of \( V(x) \). Then factor \( V(x) \) completely.
   c. Find the dimensions of the aquarium shown.

9. One special product pattern is \((a - b)^2 = a^2 - 2ab + b^2\). Using Pascal’s Triangle to expand \((a - b)^2\) gives \(1a^2 + 2a(-b) + 1(-b)^2\). Are the two expressions equivalent? Explain.

10. Can you use the synthetic division procedure that you learned in this chapter to divide any two polynomials? Explain.

11. Let \( T \) be the number (in thousands) of new truck sales. Let \( C \) be the number (in thousands) of new car sales. During a 10-year period, \( T \) and \( C \) can be modeled by the following equations where \( t \) is time (in years).
   \[
   T = 23t^4 - 330t^3 + 3500t^2 - 7500t + 9000 \\
   C = 14t^4 - 330t^3 + 2400t^2 - 5900t + 8900
   \]
   a. Find a new model \( S \) for the total number of new vehicle sales.
   b. Is the function \( S \) even, odd, or neither? Explain your reasoning.

12. Your friend has started a golf caddy business. The table shows the profits \( p \) (in dollars) of the business in the first 5 months. Use finite differences to find a polynomial model for the data. Then use the model to predict the profit after 7 months.

<table>
<thead>
<tr>
<th>Month, ( t )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profit, ( p )</td>
<td>4</td>
<td>2</td>
<td>6</td>
<td>22</td>
<td>56</td>
</tr>
</tbody>
</table>
1. The synthetic division below represents \( f(x) \div (x - 3) \). Choose a value for \( m \) so that \( x - 3 \) is a factor of \( f(x) \). Justify your answer.

\[
\begin{array}{c|ccccc}
& 3 & 1 & -3 & m & 3 \\
\hline
1 & 3 & 0 & 3 & -2 & 2 \\
1 & 0 & 0 & 0 & -1 & 1 \\
\end{array}
\]

2. Analyze the graph of the polynomial function to determine the sign of the leading coefficient, the degree of the function, and the number of real zeros. Explain.

3. About 52,300 people live in a 3-kilometer radius of a city’s center. Ten years ago, the population density in this region was about 1750 people per square kilometer. Which statement is not true?

A. The area of the region is \( 9\pi \) square kilometers.
B. The current population density is about 1850 people per square kilometer.
C. Ten years ago, there were about 49,480 people living in this region.
D. The current population density is less than it was 10 years ago.

4. A parabola passes through the point shown in the graph. The equation of the axis of symmetry is \( x = -a \). Which of the given points could lie on the parabola? If the axis of symmetry was \( x = a \), then which points could lie on the parabola? Explain your reasoning.

\[(1, 1)\] \[(0, 1)\] \[(-2, 1)\] \[(-3, 1)\] \[(-4, 1)\] \[(-5, 1)\] 

\[(3, 1)\]
5. Select values for the function to model each transformation of the graph of \( f(x) = x \).

\[ g(x) = \square (x - \square) + \square \]

a. The graph is a translation 2 units up and 3 units left.
b. The graph is a translation 2 units right and 3 units down.
c. The graph is a vertical stretch by a factor of 2, followed by a translation 2 units up.
d. The graph is a translation 3 units right and a vertical shrink by a factor of \( \frac{1}{2} \), followed by a translation 4 units down.

6. Which description represents the solid produced by rotating the figure around the given axis?

\[ \text{(A) cone with a height of 6 and a radius of 8} \]
\[ \text{(B) cone with a height of 8 and a radius of 6} \]
\[ \text{(C) pyramid with a height of 6 and a square base whose edge length is 8} \]
\[ \text{(D) pyramid with a height of 8 and a square base whose edge length is 6} \]

7. Classify each function as even, odd, or neither. Justify your answer.

\begin{align*}
\text{a. } f(x) &= 3x^5 \\
\text{b. } f(x) &= 4x^3 + 8x \\
\text{c. } f(x) &= 3x^3 + 12x^2 + 1 \\
\text{d. } f(x) &= 2x^4 \\
\text{e. } f(x) &= x^{11} - x^7 \\
\text{f. } f(x) &= 2x^8 + 4x^4 + x^2 - 5
\end{align*}

8. The volume of the rectangular prism shown is given by \( V = 2x^3 + 7x^2 - 18x - 63 \). Which polynomial represents the area of the base of the prism?

\begin{align*}
\text{(A) } 2x^2 + x - 21 \\
\text{(B) } 2x^2 + 21 - x \\
\text{(C) } 13x + 21 + 2x^2 \\
\text{(D) } 2x^2 - 21 - 13x
\end{align*}

9. The number \( R \) (in tens of thousands) of retirees receiving Social Security benefits is represented by the function \( R = 0.286t^3 - 4.68t^2 + 8.8t + 403, \ 0 \leq t \leq 10 \)

where \( t \) represents the number of years since 2000. Identify any turning points on the given interval. What does a turning point represent in this situation?

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Integrated Mathematics III
Teaching Edition

Chapter 3
Polynomial Functions
Chapter 3 Pacing Guide

<table>
<thead>
<tr>
<th>Section/Activity</th>
<th>Time</th>
</tr>
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3 Polynomial Functions

3.1 Graphing Polynomial Functions
3.2 Adding, Subtracting, and Multiplying Polynomials
3.3 Dividing Polynomials
3.4 Factoring Polynomials
3.5 Solving Polynomial Equations
3.6 The Fundamental Theorem of Algebra
3.7 Transformations of Polynomial Functions
3.8 Analyzing Graphs of Polynomial Functions
3.9 Modeling with Polynomial Functions

SEE the Big Idea

Basketball (p. 134)
Ruins of Caesarea (p. 157)
Quonset Hut (p. 174)
Zebra Mussels (p. 159)
Electric Vehicles (p. 115)

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Chapter Summary

- This is the longest chapter in the book, with nine lessons about polynomial functions. Linear and quadratic functions are two types of polynomials, so connections to earlier work are easily made.
- In the first lesson, polynomial functions are defined and graphed. The notation and vocabulary can be overwhelming for students, though some of the vocabulary was used in Math II. End behavior of even- and odd-degree polynomials is explored.
- Operations on polynomial expressions are presented so that polynomial expressions can be factored. Prior work with factoring is extended to third- and fourth-degree expressions. Synthetic division is used to efficiently check for possible rational roots when rewriting polynomials in factored form in order to solve polynomial equations.
- All of the work with operations on polynomials, factoring, and solving leads to the Fundamental Theorem of Algebra in the middle of the chapter: If \( f(x) \) is a polynomial of degree \( n \), where \( n > 0 \), then the equation \( f(x) = 0 \) has at least one solution in the set of complex numbers. The corollary to the theorem, namely that an \( n \)th-degree polynomial function has exactly \( n \) zeros, is the focus of the lesson.
- The last third of the chapter deals with polynomial functions, in particular the graphs of these functions. Earlier work with transformations is applied to polynomials. Concepts that are foundational for work in calculus are presented, and finally polynomials are used to model real-life data. Earlier work with finite differences and regression is used.
- Certainly a great deal of content in this chapter is calculator dependent. In fact, symbolic manipulators can perform much of the work presented in the early part of the chapter, and graphing calculators can be used to quickly solve polynomial equations. The balance between theory and application, and the role of technology, are decisions that mathematics departments must make not only for this chapter but for this course.

What Your Students Have Learned

Middle School
- Add, subtract, multiply, and divide rational numbers.
- Write and evaluate algebraic expressions.
- Use properties of exponents.
- Identify nonlinear functions.

Math I
- Solve linear equations.
- Use function notation to evaluate and interpret functions.
- Graph linear functions.

Math II
- Factor polynomials completely, including special products.
- Graph quadratic functions.
- Find solutions of quadratic equations.

What Your Students Will Learn

Math III
- Graph and analyze the graphs of polynomial functions, including transformations.
- Add, subtract, multiply, divide, and factor polynomials.
- Find solutions of polynomial equations and zeros of polynomial functions.
- Use the Fundamental Theorem of Algebra.
- Write polynomial functions.
Maintaining Mathematical Proficiency

Simplifying Algebraic Expressions
- Discuss ways that you can simplify an expression and how students will know whether the expression is simplified.
- Review how the Distributive Property can be used to factor or expand an expression.
- Remind students that only like terms can be combined.

**COMMON ERROR** Students may not include the sign in front of the coefficient of a term when they are simplifying.

Solving Quadratic Equations by Factoring
- Remind students to look for the greatest common factor as a first step.
- In standard form, \(ax^2 + bx + c = 0\), when \(ac\) is positive, the sign of \(b\) determines whether the factors of \(c\) are positive or negative.
- Students can use the FOIL method to check their answers.

**COMMON ERROR** Students may factor one side before they write the quadratic in standard form. Remind students to write the quadratic equation in standard form first.

Mathematical Practices (continued on page 110)
- The Mathematical Practices page focuses attention on how mathematics is learned—process versus content. Page 110 demonstrates that when using technology students must be aware of how the size of the viewing window influences what features of the function are displayed and that viewing a table of values provides additional information about the function.
- Use the Mathematical Practices page to help students develop mathematical habits of mind—how mathematics can be explored and how mathematics is thought about.

<table>
<thead>
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<th>If students got it...</th>
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<tr>
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<td>Start the next Section</td>
</tr>
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<td>Skills Review Handbook</td>
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Questioning in the Classroom
One thing leads to another.
Try to build questions from the responses given. This requires students to listen to other students’ responses.

Laurie’s Notes

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Maintaining Mathematical Proficiency

Simplifying Algebraic Expressions (Grade 7)

**Example 1** Simplify the expression $9x + 4x$.

\[ 9x + 4x = (9 + 4)x \]
\[ = 13x \]

**Example 2** Simplify the expression $2(x + 4) + 3(6 - x)$.

\[ 2(x + 4) + 3(6 - x) = 2x + 8 + 18 - 3x \]
\[ = 2x + 8 + 18 - 3x \]
\[ = 2x - 3x + 8 + 18 \]
\[ = -x + 26 \]

Simplify the expression.

1. $6x - 4x$
2. $12m - 4m - 7m + 3$
3. $3(y + 2) - 4y$
4. $9x - 4(2x - 1)$
5. $-2(z + 2) - 2(1 - z)$
6. $-x^2 + 5x + x^2$

Solving Quadratic Equations by Factoring (Math II)

**Example 3** Solve $x^2 + 7x = 18$.

\[ x^2 + 7x - 18 = 0 \]
\[ (x + 9)(x - 2) = 0 \]
\[ x + 9 = 0 \quad \text{or} \quad x - 2 = 0 \]
\[ x = -9 \quad \text{or} \quad x = 2 \]

The solutions are $x = -9$ and $x = 2$.

Solve the equation by factoring.

7. $x^2 + 3x + 2 = 0$
8. $x^2 - 6x + 8 = 0$
9. $x^2 + 10x = -25$
10. $2x^2 - 84 = 2x$
11. $4x^2 = 12x - 9$
12. $8x - 3 = -3x^2$
13. ABSTRACT REASONING Explain how you can find the solutions of an equation of the form $(x - a)(x - b)(x - c) = 0$.

What Your Students Have Learned

- Simplify algebraic expressions using the Commutative, Associative, and Distributive Properties and the order of operations.
- Solve quadratic equations by factoring and using the Zero-Product Property.

ANSWERS

1. $2x$
2. $4m + 3$
3. $-y + 6$
4. $x + 4$
5. $z - 4$
6. $5s$
7. $x = -1, x = -2$
8. $x = 2, x = 4$
9. $x = -5$
10. $x = -6, x = 7$
11. $x = \frac{3}{2}$
12. $x = -3, x = \frac{1}{3}$
13. Sample answer: use the Zero-Product Property to set each of the factors equal to zero and solve for $x$.

Vocabulary Review

Have students make a Four Square for each topic.

- Simplifying algebraic expressions
- Solving quadratic equations by factoring

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Continuous Functions

A function is continuous when its graph has no breaks, holes, or gaps.

Core Concept

Mathematically proficient students use technological tools to explore concepts.

Monitoring Progress

Use a graphing calculator to determine whether the function is continuous. Explain your reasoning.

1. \( f(x) = x^2 - x - x^2 \)
2. \( f(x) = x^3 - 3 \)
3. \( f(x) = \sqrt{x^2 + 1} \)
4. \( f(x) = |x + 2| \)
5. \( f(x) = \frac{1}{x} \)
6. \( f(x) = \frac{1}{\sqrt{x^2 - 1}} \)
7. \( f(x) = x \)
8. \( f(x) = 2x - 3 \)
9. \( f(x) = \frac{x}{a} \)

Example 1

Determining Whether Functions Are Continuous

Use a graphing calculator to compare the two functions. What can you conclude? Which function is not continuous?

\[ f(x) = x^2 \]
\[ g(x) = \frac{x^3 - x^2}{x - 1} \]

**SOLUTION**

The graphs appear to be identical, but \( g(x) \) is not defined when \( x = 1 \). There is a hole in the graph of \( g(x) \) at the point (1, 1). Using the table feature of a graphing calculator, you obtain an error for \( g(x) \) when \( x = 1 \). So, \( g(x) \) is not continuous.

Mathematical Practices

(continued from page T-109)

- Example 1 shows the graphs of two functions that appear to be the same with the exception of a hole in the graph. The hole appears when you zoom to a decimal viewing window (\( \Delta x = 0.1 \)).
- Looking at the table of values reveals that there is an error at \( x = 1 \).
- The graph of the quadratic \( y = x^2 \) is continuous over the whole domain. Examining the rational function with technology reveals that it is not continuous.
- Students could work with partners on Monitoring Progress. If the function is not continuous, then students should note the window size and the \( x \)-value where the function is not continuous.
### Core Vocabulary

- Polynomial, p. 112
- Polynomial function, p. 112
- End behavior, p. 113
- Pascal’s Triangle, p. 123
- Binomial Theorem, p. 124
- Polynomial long division, p. 130
- Synthetic division, p. 131
- Factored completely, p. 136
- Factor by grouping, p. 137
- Quadratic form, p. 137

### Core Concepts

#### Section 3.1
- Common Polynomial Functions, p. 112
- End Behavior of Polynomial Functions, p. 113
- Graphing Polynomial Functions, p. 114

#### Section 3.2
- Operations with Polynomials, p. 120
- Special Product Patterns, p. 121
- Pascal’s Triangle, p. 123
- The Binomial Theorem, p. 124

#### Section 3.3
- Polynomial Long Division, p. 130
- Synthetic Division, p. 131
- The Remainder Theorem, p. 132

#### Section 3.4
- Factoring Polynomials, p. 136
- Special Factoring Patterns, p. 136
- The Factor Theorem, p. 138

### Mathematical Practices

1. Describe the entry points you used to analyze the function in Exercise 43 on page 118.

2. Describe how you maintained oversight in the process of factoring the polynomial in Exercise 49 on page 141.

### Keeping Your Mind Focused

- When you sit down at your desk, review your notes from the last class.
- Repeat in your mind what you are writing in your notes.
- When a mathematical concept is particularly difficult, ask your teacher for another example.

### ANSWERS

1. Understanding that the statement \( g(x) = -f(x) \) is a reflection in the \( x \)-axis allows you to use reason to determine the end behavior of the reflected function.

2. When doing long or synthetic division, always check to see if there are any terms with zeros as coefficients. In this problem, the exponents are 3, 0, and 1, so when doing synthetic division, a coefficient of 0 should be included for the quadratic term.
ANSWERS

1. polynomial function;  
   \[ f(x) = -3x^4 - x^3 + 2x^2 - 2x + 5; \]
   degree: 4 (quartic), leading coefficient: -3

2. polynomial function;  
   \[ g(x) = \frac{x}{3} + 3x^2 + 2x + 1; \]
   degree: 3 (cubic), leading coefficient: \( \frac{1}{3} \)

3. not a polynomial function

4. a. The function is increasing when  
   \[ x < 2 \] and decreasing when \( x > 2 \).
   b. \( 1 < x < 3 \)
   c. \( x < 1 \) and \( x > 3 \)

5. The area is 3 \( x^2 + 7x + 3 \) and the perimeter is 8 \( x + 8 \).

6. 4 \( x^2 + 5x - 5 \)

7. 3 \( x^3 - 10x^2 + 9x - 2 \)

8. \( x^3 - 2x^2 - 11x + 12 \)

9. \( x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32 \)

10. 4 \( x^2 + 6x + 17 + \frac{35x + 25}{x^2 - 2x - 1} \)

11. \( a(a - 4)(a + 2) \)

12. \( (2m + 3)(4m^2 - 6m + 9) \)

13. \( (z - 2)(z^2 + 2z - 1) \)

14. \( (7b^2 - 8)(7b^2 + 8) \)

15. -5

16. a. \( f(x) = (x + 5)(x - 3)(x - 4) \)

b. 0.6 cents per year

17. The model makes sense for  
   \( x > 4 \). In factored form, the volume is  \( V(x) = x(2x - 3)(x - 4) \). For all three dimensions of the crate to be positive, \( x \) must always be greater than 4.

3.1–3.4 Quiz

Decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient. (Section 3.1)

1. \( f(x) = 5 + 2x^2 - 3x^4 - 2x - x \)

2. \( g(x) = \frac{1}{3}x^3 + 2x - 3x^2 + 1 \)

3. \( h(x) = 3 - 6x^3 + 4x^2 + 6x \)

4. Describe the \( x \)-values for which (a) \( f \) is increasing or decreasing, (b) \( f(x) > 0 \), and (c) \( f(x) < 0 \). (Section 3.1)

5. Write an expression for the area and perimeter for the figure shown. (Section 3.2)

6. \( (7x^2 - 4) - (3x^2 - 5x + 1) \)

7. \( (x^2 - 3x + 2)(3x - 1) \)

8. \( (x - 1)(x + 3)(x - 4) \)

9. Use Pascal’s Triangle or the Binomial Theorem to expand \( (x + 2)^3 \). (Section 3.2)

10. Divide \( 4x^4 - 2x^3 + x^2 - 5x + 8 \) by \( x^2 - 2x - 1 \). (Section 3.3)

Factor the polynomial completely. (Section 3.4)

11. \( a^2 - 2a^2 - 8a \)

12. \( 8a^3 + 27 \)

13. \( z^3 + z^2 - 4z - 4 \)

14. \( 4b^4 - 64 \)

15. Show that \( x + 5 \) is a factor of \( f(x) = x^3 - 2x^2 - 23x + 60 \). Then factor \( f(x) \) completely. (Section 3.4)

16. The estimated price \( P \) (in cents) of stamps in the United States can be modeled by the polynomial function \( P(t) = 0.007t^3 - 0.16t^2 + 17t + 1990 \), where \( t \) represents the number of years since 1990. (Section 3.1)
   a. Use a graphing calculator to graph the function for the interval \( 0 \leq t \leq 20 \).
   b. What was the average rate of change in the price of stamps from 1990 to 2010?

17. The volume \( V \) (in cubic feet) of a rectangular wooden crate is modeled by the function  
   \( V(x) = 2x^3 - 11x^2 + 12x \), where \( x \) is the width (in feet) of the crate. Determine the values of \( x \) for which the model makes sense. Explain your reasoning. (Section 3.4)
Laurie’s Notes

Overview of Section 3.8

Introduction
• The exploration and lesson pull together many of the skills and concepts of the chapter in order to analyze graphs of polynomial functions. The lesson begins with a summary of vocabulary related to zeros of a function, followed by the Location Principle. This principle is a numeric approach to finding zeros of a function.
• Turning points are introduced in the exploration. The maximum number of turning points for a function of degree \( n \) follows from the exploration.
• The last topic related to analyzing the graphs of polynomial functions is even and odd functions. When it is known that a function is even (or odd), only half of the graph needs to be plotted. For instance, graph the function for \( x \geq 0 \). When the function is even (or odd), the remaining graph \( (x < 0) \) can be sketched quickly.

Common Misconceptions
• Students may confuse a point of inflection with a turning point. This can be particularly true for certain viewing windows.
• The cubic \( f(x) = 0.25x^3 \) shown is an increasing function over all values of the domain. The origin \((0, 0)\) is a point of inflection where the graph changes from concave down to concave up.

• Remind students that the function must change from being increasing to decreasing, or vice versa, to be a turning point.

Formative Assessment Tips
• Point of Most Significance: This technique asks students to identify the most significant idea, learning, or concept they gained in the lesson today. Students reflect on the lesson and are asked to identify the key example, problem, or point that was significant in their learning today. If the goal or learning intention was identified at the beginning of the class, then students assess what contributed to their attainment of the goal.
• It is important for teachers to know whether the lesson as designed helped move learning forward for students or whether the lesson needs to be modified. Share with students what you learned from their reflections. Students will take reflections more seriously when they are valued and used.

Pacing Suggestion
• As students work through the explorations, listen and probe for recall of prior skills. What evidence of reasoning do you hear? Continue with the formal lesson.
What Your Students Will Learn

- Identify and use x-intercepts to graph polynomial functions.
- Use the Location Principle to find real zeros of polynomial functions.
- Find turning points of graphs of polynomial functions.
- Identify local maximums and local minimums of graphs of polynomial functions.
- Identify even and odd functions algebraically and graphically.

Exploration

Motivate

? "What do the letters A, H, I, M, O, T, V, W, X, and Y have in common?" They all have a vertical line of symmetry.

? "What do the letters H, I, O, S, X, and Z have in common?" They all have a rotational symmetry of 180°.

- Explain to students that today they will look at functions that have symmetry about the y-axis and rotational symmetry about the origin.

Exploration 1

- Reason Abstractly and Quantitatively: Consider what reasoning students must use to match the equation with the graph. Two of the equations are quadratic, both opening upward, but one has a positive y-intercept and the other has a negative y-intercept. There is only one fourth-degree polynomial, and the end behavior matches the graph. There is one cubic with a leading coefficient that is positive, and it matches the end behavior of a cubic graph. That leaves two odd-degree polynomials with leading coefficients that are negative. One of these polynomials has a positive y-intercept, and the other has a negative y-intercept.
- This would be the expected level of reasoning from students at this point in the chapter and course of study.
- If students are not familiar with finding the local maximum and minimum values of a function, then you may need to review this.
- Attend to Precision: Note that the directions specify that coordinates for turning points should be rounded to the nearest hundredth.

Communicate Your Answer

- Construct Viable Arguments and Critique the Reasoning of Others: Ask students to explain their reasoning for each question.
- Extension: Ask students whether it is possible to sketch the graph of a quadratic or quartic that has no turning points.

Connecting to Next Step

- Students should have a good sense of different characteristics of even- and odd-degree polynomials at this point. Their investigation of turning points provides a good foundation for the middle portion of the formal lesson.
3.8 Analyzing Graphs of Polynomial Functions

Essential Question: How many turning points can the graph of a polynomial function have?

A turning point of the graph of a polynomial function is a point on the graph at which the function changes from
• increasing to decreasing, or
• decreasing to increasing.

Approximating Turning Points

Work with a partner. Match each polynomial function with its graph. Explain your reasoning. Then use a graphing calculator to approximate the coordinates of the turning points of the graph of the function. Round your answers to the nearest hundredth.

a. \( f(x) = 2x^2 + 3x - 4 \)

b. \( f(x) = x^3 + 3x + 2 \)

c. \( f(x) = x^3 - 2x^2 - x + 1 \)

d. \( f(x) = -x^3 + 5x - 2 \)

e. \( f(x) = x^4 - 3x^3 + 2x - 1 \)

Communicate Your Answer

2. How many turning points can the graph of a polynomial function have?

3. Is it possible to sketch the graph of a cubic polynomial function that has no turning points? Justify your answer.

Section 3.8 Analyzing Graphs of Polynomial Functions 167

Answers

1. a. C; \( f \) is a quadratic function with a y-intercept of \(-4\); \((0.75, -5.13)\)

b. F; \( f \) is a quadratic function with a y-intercept of \(2\); \((-1.50, -0.25)\)

c. A; \( f \) is a cubic function with a y-intercept of \(1\); \((1.53, -1.63), (-0.22, -0.25)\)

d. D; \( f \) is a cubic function with a y-intercept of \(-2\); \((-1.29, -6.30), (1.29, 2.30)\)

e. B; \( f \) is a quartic function; \((1, 1), (0.37, -0.65), (-1.34, -5.84)\)

f. E; \( f \) is a fifth degree function; \((-0.89, -1.13), (0.77, 5.72)\)

2. The graph of every polynomial function of degree \(n\) has at most \((n - 1)\) turning points.

3. yes; \( f(x) = x^3 \) has no turning points because its graph is always increasing.
What You Will Learn

- Use x-intercepts to graph polynomial functions.
- Use the Location Principle to identify zeros of polynomial functions.
- Find turning points and identify local maximums and local minimums of graphs of polynomial functions.
- Identify even and odd functions.

Graphing Polynomial Functions

In this chapter, you have learned that zeros, factors, solutions, and x-intercepts are closely related concepts. Here is a summary of these relationships.

Core Concept

Zeros, Factors, Solutions, and Intercepts

Let \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \) be a polynomial function.

- Zero: \( k \) is a zero of the polynomial function \( f \).
- Factor: \( x - k \) is a factor of the polynomial \( f(x) \).
- Solution: \( k \) is a solution (or root) of the polynomial equation \( f(x) = 0 \).
- x-Intercept: If \( k \) is a real number, then \( k \) is an x-intercept of the graph of the polynomial function \( f \). The graph of \( f \) passes through \((k, 0)\).

Example 1

Using x-Intercepts to Graph a Polynomial Function

Graph the function
\[ f(x) = \frac{1}{4}(x + 3)(x - 2)^2. \]

**SOLUTION**

**Step 1** Plot the x-intercepts. Because \(-3\) and \(2\) are zeros of \( f \), plot \((-3, 0)\) and \((2, 0)\).

**Step 2** Plot points between and beyond the x-intercepts.

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<th>0</th>
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<tr>
<td>( y )</td>
<td>(-1)</td>
<td>(2)</td>
<td>(\frac{1}{4})</td>
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</table>

Step 3 Determine end behavior. Because \( f(x) \) has three factors of the form \( x - k \) and a constant factor of \( \frac{1}{4} \), \( f \) is a cubic function with a positive leading coefficient. So, \( f(x) \to \infty \) as \( x \to -\infty \) and \( f(x) \to \infty \) as \( x \to +\infty \).

Step 4 Draw the graph so that it passes through the plotted points and has the appropriate end behavior.

**Monitoring Progress**

Graph the function.
1. \( f(x) = \frac{1}{4}(x + 1)(x - 4)^2 \)
2. \( f(x) = \frac{1}{2}(x + 2)(x - 1)(x - 3) \)

Laurie’s Notes

- Thumbs Up: Vocabulary that may have been used almost interchangeably during the chapter is summarized in the Concept Summary. Have students give a Thumbs Up self-assessment of the Concept Summary.

“Do what you do know about the graph of \( f(x) \)? Explain.” The x-intercepts are \(-3\) and \(2\), and \(2\) is a double root. The end behavior is \((\infty, \infty)\). The coordinates of the y-intercept are \((0, 2)\). Plot the known information and sketch an approximate graph.

- Think-Pair-Share: Have students work independently on Questions 1 and 2 and then share with partners. Discuss as a class.
**The Location Principle**

You can use the **Location Principle** to help you find real zeros of polynomial functions.

**Core Concept**

**The Location Principle**

If \( f \) is a polynomial function, and \( a \) and \( b \) are two real numbers such that \( f(a) < 0 \) and \( f(b) > 0 \), then \( f \) has at least one real zero between \( a \) and \( b \).

To use this principle to locate real zeros of a polynomial function, find a value \( a \) at which the polynomial function is negative and another value \( b \) at which the function is positive. You can conclude that the function has at least one real zero between \( a \) and \( b \).

**EXAMPLE 2**

**Locating Real Zeros of a Polynomial Function**

Find all real zeros of

\[ f(x) = 6x^4 + 5x^2 - 17x - 6. \]

**SOLUTION**

**Step 1** Use a graphing calculator to make a table.

**Step 2** Use the Location Principle. From the table shown, you can see that \( f(1) < 0 \) and \( f(2) > 0 \). So, by the Location Principle, \( f \) has a zero between 1 and 2. Because \( f \) is a polynomial function of degree 3, it has three zeros. The only possible rational zeros between 1 and 2 is \( \frac{3}{2} \). Using synthetic division, you can confirm that \( \frac{3}{2} \) is a zero.

**Step 3** Write \( f(x) \) in factored form. Dividing \( f(x) \) by its known factor \( x - \frac{3}{2} \) gives a quotient of \( 6x^3 + 14x + 4 \). So, you can factor \( f(x) \) as

\[
 f(x) = \left(x - \frac{3}{2}\right)(6x^3 + 14x + 4) \\
 = 2\left(x - \frac{3}{2}\right)(3x^2 + 7x + 2) \\
 = 2\left(x - \frac{3}{2}\right)(3x + 1)(x + 2).
\]

From the factorization, there are three zeros. The zeros of \( f \) are

\[ \frac{3}{2}, -\frac{1}{3}, \text{ and } -2. \]

**Check**

Check this by graphing \( f \).

**Monitoring Progress**

3. Find all real zeros of \( f(x) = 18x^3 + 21x^2 - 13x - 6. \)

---

**Laurie’s Notes**

- **Question:** If a continuous function passes through these two points, does the function have any zeros? Explain.
  - **Answer:** Yes; The graph must cross the x-axis between 1 and 2.

- **State the Location Principle, which makes sense to students.**

- **Another Way:** Write the equation in Example 2 and display a table of function values from -3 to 3.

- **Turn and Talk:** "Explain to your partners everything you know about the graph of the function." Have students verify their conjectures by graphing the function on a calculator.

---

**Differentiated Instruction**

**Auditory**

Read aloud the Core Concept: Location Principle in its entirety. Then read it again, but break the sentence down into smaller segments. Ask students to summarize these segments as you read them aloud. Then, display the graph shown under the Core Concept and have students describe how it models the Location Principle.

**Extra Example 2**

Find all real zeros of \( f(x) = 4x^3 + 28x^2 + 21x - 18. \)

The zeros of \( f \) are \(-6, \frac{1}{2}, \) and \(-\frac{1}{2}.\)

**MONITORING PROGRESS**

ANSWER

\[ 3, -\frac{1}{2}, \frac{1}{3}, \text{ and } \frac{5}{3}. \]

---

**Section 3.8  Analyzing Graphs of Polynomial Functions  169**
Extra Example 3
Graph each function. Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which each function is increasing or decreasing.

a. \(f(x) = x^3 - 2x^2 + x - 2\)

The x-intercept of the graph is \(x = 2\). The function has a local maximum at \((0.33, -1.85)\) and a local minimum at \((1, -2)\). The function is increasing when \(x < 0.33\) and \(x > 1\) and decreasing when \(0.33 < x < 1\).

b. \(f(x) = x^4 - 5x^3 + 2x^2 + 8x - 2\)

The x-intercepts of the graph are \(x = -1.11, x = 0.24, x = 1.82\), and \(x = 4.05\). The function has a local maximum at \((1.07, 4.04)\) and local minimums at \((-0.57, -4.88)\) and \((3.25, -14.95)\). The function is increasing when \(-0.57 < x < 1.07\) and \(x > 3.25\) and decreasing when \(x < -0.57\) and \(1.07 < x < 3.25\).

MONITORING PROGRESS
ANSWER

4.

The x-intercept of the graph is \(x = -3.07\). The function has a local maximum at \((-1.72, 4.13)\) and a local minimum at \((0.39, 1.79)\). The function is increasing when \(-1.72 < x < 0.39\) and decreasing when \(-1.72 < x < 0.39\).

Turning Points

Another important characteristic of graphs of polynomial functions is that they have turning points corresponding to local maximum and minimum values.

- The x-coordinate of a turning point is a local maximum of the function when the point is higher than all nearby points.
- The x-coordinate of a turning point is a local minimum of the function when the point is lower than all nearby points.

The turning points of a graph help determine the intervals for which a function is increasing or decreasing. You can write these intervals using interval notation.

Core Concept

Turning Points of Polynomial Functions

1. The graph of every polynomial function of degree \(n\) has at most \(n - 1\) turning points.
2. If a polynomial function has \(n\) distinct real zeros, then its graph has exactly \(n - 1\) turning points.

EXAMPLE 3
Finding Turning Points

Graph each function. Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which each function is increasing or decreasing.

a. \(f(x) = x^3 - 3x^2 + 6\)

SOLUTION

a. Use a graphing calculator to graph the function. The graph of \(f\) has one x-intercept and two turning points. Use the graphing calculator’s zero, maximum, and minimum features to approximate the coordinates of the points.

The x-intercept of the graph is \(x = -1.20\). The function has a local maximum at \((0.6, 6)\) and a local minimum at \((2, 2)\). The function is increasing when \(x < 0\) and \(x > 2\) and decreasing when \(0 < x < 2\).

b. Use a graphing calculator to graph the function. The graph of \(g\) has four x-intercepts and three turning points. Use the graphing calculator’s zero, maximum, and minimum features to approximate the coordinates of the points.

The x-intercepts of the graph are \(x = -1.14, x = 0.29, x = 1.82\), and \(x = 5.03\). The function has a local maximum at \((1.11, 5.11)\) and local minimums at \((-0.57, -6.50)\) and \((3.96, -43.04)\). The function is increasing when \(-0.57 < x < 1.11\) and \(x > 3.96\) and decreasing when \(-0.57 < x < 0.29\) and \(1.11 < x < 3.96\).

Monitoring Progress

4. Graph \(f(x) = 0.5x^3 + x^2 - x + 2\). Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which the function is increasing or decreasing.

Laurie’s Notes

- Discuss findings from the exploration. Write the Core Concept.
- “Why do you think a turning point is referred to as a local or relative maximum?” Answers will vary. Listen for understanding that the function may indeed have a greater function value away from the turning point. Locally, it is a maximum value.
- Think-Pair-Share: Have students work independently on the two functions in Example 3.
- Use Appropriate Tools Strategically: Students should demonstrate an ability to use their calculators effectively in finding the roots and turning points.
Even and Odd Functions

**Core Concept**

**Even and Odd Functions**

A function $f$ is an **even function** when $f(-x) = f(x)$ for all $x$ in its domain. The graph of an even function is **symmetric about the y-axis**.

A function $f$ is an **odd function** when $f(-x) = -f(x)$ for all $x$ in its domain. The graph of an odd function is **symmetric about the origin**. One way to recognize a graph that is symmetric about the origin is that it looks the same after a 180° rotation about the origin.

![Graphs of Even and Odd Functions](image)

For an even function, if $(x, y)$ is on the graph, then $(-x, y)$ is also on the graph.

For an odd function, if $(x, y)$ is on the graph, then $(-x, -y)$ is also on the graph.

**EXAMPLE 4**

**Identifying Even and Odd Functions**

Determine whether each function is even, odd, or neither.

**SOLUTION**

a. $f(x) = x^3 - 7x$

b. $g(x) = x^4 + x^2 - 1$

c. $h(x) = x^2 + 2$

**Extra Example 4**

Determine whether each function is even, odd, or neither.

a. $f(x) = x^3 - 12x$ **odd**

b. $g(x) = x^3 + x^2 - 7$ **even**

c. $h(x) = x^5 + 9$ **neither**

**MONITORING PROGRESS**

**Answers**

5. **even**

6. **neither**

7. **odd**

---

**Laurie’s Notes**

**Teacher Actions**

**COMMON ERROR** A polynomial with an even degree (or an odd degree) does not make it an even function (or an odd function). Discuss the definitions.

"How can you test whether a function is even, odd, or neither?" Replace $x$ with $-x$ in the function and simplify. The result will be $f(x)$ [even], $-f(x)$ [odd], or something else [neither].

**Closure**

- **Point of Most Significance**: Ask students to identify, aloud or on a paper to be collected, the most significant point (or part) in the lesson that aided their learning.
ANSWERS
1. turning
2. A local maximum is a turning point of a graph where the y-coordinate is higher than all nearby points. It is different from the maximum value of a function because it may not be the highest point on the entire graph.
3. A
4. C
5. B
6. D
7. In Exercises 7–14, graph the function. (See Example 1.)
   7. \(f(x) = (x - 2)(x + 1)\)
   8. \(f(x) = (x + 2)(x + 4)\)
   9. \(h(x) = (x + 1)(x - 1)(x - 3)\)
   10. \(g(x) = 4(x + 1)(x - 2)(x - 1)\)
   11. \(l(x) = \frac{1}{2}(x - 5)(x + 2)(x - 3)\)
   12. \(g(x) = \frac{1}{2}(x + 4)(x + 8)(x - 1)\)
   13. \(h(x) = (x - 3)(x^2 + x + 1)\)
   14. \(f(x) = (x - 4)(2x^2 - 2x + 1)\)

8. In Exercises 15 and 16, describe and correct the error in using factors to graph \(f\). 
   15. \(f(x) = (x + 2(x - 1)^2\)
   16. \(f(x) = x^2(x - 3)^2\)

9. In Exercises 17–22, find all real zeros of the function. (See Example 2.)
   17. \(f(x) = x^3 - 4x^2 - x + 4\)
   18. \(f(x) = x^3 - 3x^2 - 4x + 12\)
   19. \(h(x) = 2x^2 - 7x^2 - 5x - 4\)
   20. \(h(x) = 4x^3 - 2x - 24x - 18\)
   21. \(g(x) = 4x^3 + x^2 - 5x + 36\)
   22. \(f(x) = 2x^3 - 3x^2 - 32x - 15\)

10. \(f(x) = (x - 1)(x + 2)(x + 3)(x + 4)\)

11. \(f(x) = (x + 1)(x + 2)(x + 3)(x + 4)\)

12–22. See Additional Answers.
In Exercises 23–30, graph the function. Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which the function is increasing or decreasing. (See Example 3.)

23. \( g(x) = 2x^4 + 8x^2 - 3 \)
24. \( g(x) = -x^4 + 3x \)
25. \( h(x) = x^4 - 3x^2 + x \)
26. \( f(x) = x^4 - 4x^3 + x^2 + 2 \)
27. \( f(x) = 0.7x^3 - 3x^2 + 5x \)
28. \( h(x) = x^3 + 2x^2 - 17x - 4 \)
29. \( g(x) = x^4 - 5x^3 + 2x^2 + x - 3 \)
30. \( g(x) = x^4 - 5x^2 + 2x^2 + x - 3 \)

In Exercises 31–36, estimate the coordinates of each turning point. State whether each corresponds to a local maximum or a local minimum. Then estimate the real zeros and find the least possible degree of the function.

31. 
32. 
33. 
34. 
35. 
36. 

OPEN-ENDED In Exercises 37 and 38, sketch a graph of a polynomial function \( f \) having the given characteristics.

37. • The graph of \( f \) has \( x \)-intercepts at \( x = -4, x = 0, \) and \( x = 2. \)
   • \( f \) has a local maximum value when \( x = 1. \)
   • \( f \) has a local minimum value when \( x = -2. \)

38. • The graph of \( f \) has \( x \)-intercepts at \( x = -3, x = 1, \) and \( x = 5. \)
   • \( f \) has a local maximum value when \( x = 1. \)
   • \( f \) has a local minimum value when \( x = -2 \) and when \( x = 4. \)

In Exercises 39–46, determine whether the function is even, odd, or neither. (See Example 4.)

39. \( h(x) = 4x^2 \)
40. \( g(x) = -2x^6 + x^2 \)
41. \( f(x) = x^4 + 3x^2 - 2 \)
42. \( f(x) = x^4 + 3x^3 - x \)
43. \( g(x) = x^2 + 5x + 1 \)
44. \( f(x) = -x^4 + 2x - 9 \)
45. \( f(x) = x^4 - 12x^2 \)
46. \( h(x) = x^4 + 3x^4 \)

47. USING TOOLS When a swimmer does the breaststroke, the function

\[
S = -241t^2 + 1060t - 1870t^3 + 1650t^4 - 733t^5 + 144t^7 - 2.43t
\]

models the speed \( S \) (in meters per second) of the swimmer during one complete stroke, where \( t \) is the number of seconds since the start of the stroke and \( 0 \leq t \leq 1.22 \). Use a graphing calculator to graph the function. At what time during the stroke is the swimmer traveling the fastest?

48. USING TOOLS During a recent period of time, the number \( S \) (in thousands) of students enrolled in public schools in a certain country can be modeled by

\[
S = 1.64t^4 - 102t^3 + 1710t^2 + 36,300,
\]

where \( t \) is time (in years). Use a graphing calculator to graph the function for the interval \( 0 \leq t \leq 41 \). Then describe how the public school enrollment changes over this period of time.

49. WRITING Why is the adjective local, used to describe the maximums and minimums of cubic functions, sometimes not required for quadratic functions?

Section 3.8 Analyzing Graphs of Polynomial Functions

27–38. See Additional Answers.

39. odd
40. even
41. even
42. odd
43. neither
44. neither
45. even
46. neither
47. 

about 1 second into the stroke

48. See Additional Answers.

49. A quadratic function only has one turning point, and it is always the maximum or minimum value of the function.

23. 

The \( x \)-intercepts of the graph are \( x = -3.90, x = -0.67, \) and \( x = 0.57. \) The function has a local maximum at \((-2.67, 15.96)\) and a local minimum at \((0, -3)\); The function is increasing when \( x < -2.67 \) and \( x > 0 \) and is decreasing when \(-2.67 < x < 0.\)

24. 

The \( x \)-intercepts of the graph are \( x = 0 \) and \( x = 1.44. \) The function has a local maximum at \((0.91, 2.04)\); The function is increasing when \( x < 0.91 \) and is decreasing when \( x > 0.91.\)

25. 

The \( x \)-intercepts of the graph are \( x = -1.88, x = 0, x = 0.35, \) and \( x = 1.53. \) The function has a local maximum at \((0.17, 0.08)\) and local minimums at \((-1.30, -3.51)\) and \((-1.13, -1.07); The function is increasing when \(-1.30 < x < 0.17 \) and \( x > 1.13 \) and is decreasing when \( x < -1.30 \) and \( 0.17 < x < 1.13.\)

26. 

The \( x \)-intercepts of the graph are \( x = -2.16, x = 1, \) and \( x = 1.75. \) The function has a local maximum at \((-1.63, 10.47)\) and a local minimum at \((1.46, -1.68); The function is increasing when \( x < -1.63 \) and \( x > 1.46 \) and is decreasing when \(-1.63 < x < 1.46.\)

Section 3.8

173
ANSWERS
50. a. The zeros of the function are 
−3 and 0. The local maximum is 
(−2, 4) and the local minimum is 
(0, 0).
b. The x-intercepts of the graphs of 
y = f(x) and y = −f(x) are the same.
c. The minimum value of y = f(x) is 
the opposite of the maximum value of 
y = −f(x) and the maximum value of y = f(x) is 
the opposite of the minimum value of 
y = −f(x).
51–57. See Additional Answers.

Mini-Assessment
1. Graph the function

\[ f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 4x. \]

2. Find all real zeros of

\[ f(x) = 3x^3 - 11x^2 - 6x + 8. \]

−1, 4, \frac{1}{3}

3. Graph the function

\[ f(x) = x^3 - 2x^2 + x - 4. \] Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which the function is increasing or decreasing.

The x-intercept of the graph is 
x = 2.31. The function has a local maximum at (0.33, −3.85) and a local minimum at (1, −4). The function is increasing when x < 0.33 and x > 1. It is decreasing when 
0.33 < x < 1.

Determine whether each function is even, odd, or neither.
4. \( g(x) = 20x^4 + 8x^3 - 2x \) odd
5. \( h(x) = 9x^4 + 5x^3 + 3 \) even

50. HOW DO YOU SEE IT? The graph of a polynomial function is shown.

a. Find the zeros, local maximum, and 
local minimum values of the function.
b. Compare the x-intercepts of the graphs of 
y = f(x) and y = −f(x).
c. Compare the maximum and minimum values of 
the functions y = f(x) and y = −f(x).

51. MAKING AN ARGUMENT Your friend claims that the product of two odd functions is an odd function. Is your friend correct? Explain your reasoning.

52. MODELING WITH MATHEMATICS You are making a rectangular box out of a 16-inch-by-20-inch piece of cardboard. The box will be formed by making the cuts shown in the diagram and folding up the sides. You want the box to have the greatest volume possible.

a. How long should you make the cuts?
b. What is the maximum volume?
c. What are the dimensions of the finished box?

53. PROBLEM SOLVING Quonset huts are temporary, all-purpose structures shaped like half-cylinders. You have 1100 square feet of material to build a quonset hut.
a. The surface area S of a quonset hut is given by 
\[ S = 2\pi r^2 + \pi r h. \] Substitute 1100 for S and then 
write an expression for R in terms of r.
b. The volume V of a quonset hut is given by 
\[ V = \frac{1}{2}\pi r^2 h. \] Write an equation that gives V as a function in terms of r only.
c. Find the value of r that maximizes the volume of the hut.

54. THOUGHT PROVOKING Write and graph a polynomial function that has one real zero in each of the intervals 
−2 < x < −1, 0 < x < 1, and 4 < x < 5. Is there a maximum degree that such a polynomial function can have? Justify your answer.

55. MATHEMATICAL CONNECTIONS A cylinder is inscribed in a sphere of radius 8 inches. Write an 
equation for the volume of the cylinder as a function of h. Find the value of h that maximizes the volume of 
the inscribed cylinder. What is the maximum volume of the cylinder?

56. State whether the table displays linear data, quadratic data, or neither. Explain. (Section 2.7)

<table>
<thead>
<tr>
<th>Months, x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Savings (dollars, y)</td>
<td>100</td>
<td>150</td>
<td>200</td>
<td>250</td>
</tr>
</tbody>
</table>

57. Time (seconds), x | 0 | 1 | 2 | 3 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (feet), y</td>
<td>300</td>
<td>284</td>
<td>236</td>
<td>156</td>
</tr>
</tbody>
</table>

53. PROBLEM SOLVING Quonset huts are temporary, all-purpose structures shaped like half-cylinders. You have 1100 square feet of material to build a quonset hut.
a. The surface area S of a quonset hut is given by 
\[ S = 2\pi r^2 + \pi r h. \] Substitute 1100 for S and then 
write an expression for R in terms of r.
b. The volume V of a quonset hut is given by 
\[ V = \frac{1}{2}\pi r^2 h. \] Write an equation that gives V as a function in terms of r only.
c. Find the value of r that maximizes the volume of the hut.

54. THOUGHT PROVOKING Write and graph a polynomial function that has one real zero in each of the intervals 
−2 < x < −1, 0 < x < 1, and 4 < x < 5. Is there a maximum degree that such a polynomial function can have? Justify your answer.

55. MATHEMATICAL CONNECTIONS A cylinder is inscribed in a sphere of radius 8 inches. Write an 
equation for the volume of the cylinder as a function of h. Find the value of h that maximizes the volume of 
the inscribed cylinder. What is the maximum volume of the cylinder?
3.5–3.9 What Did You Learn?

Core Vocabulary
- repeated solution, p. 146
- complex conjugates, p. 155
- local maximum, p. 170
- local minimum, p. 170
- finite differences, p. 176
- even function, p. 171
- odd function, p. 171

Core Concepts
- Section 3.5
  - The Rational Root Theorem, p. 147
  - The Irrational Conjugates Theorem, p. 149
- Section 3.6
  - The Fundamental Theorem of Algebra, p. 154
  - Descartes’s Rule of Signs, p. 156
  - The Complex Conjugates Theorem, p. 155
- Section 3.7
  - Transformations of Polynomial Functions, p. 162
  - Writing Transformed Polynomial Functions, p. 163
- Section 3.8
  - Zeros, Factors, Solutions, and Intercepts, p. 168
  - Turning Points of Polynomial Functions, p. 170
  - The Location Principle, p. 169
  - Even and Odd Functions, p. 171
- Section 3.9
  - Writing Polynomial Functions for Data Sets, p. 176
  - Properties of Finite Differences, p. 177

Mathematical Practices
1. Explain how understanding the Complex Conjugates Theorem allows you to construct your argument in Exercise 46 on page 159.
2. Describe how you use structure to accurately match each graph with its transformation in Exercises 7–10 on page 165.

Performance Task:

Quonset Huts

Over 153,000 Quonset huts were procured by the United States Navy during the 1940s. The most common huts were 20 feet wide and 48 feet long. How many different sizes of Quonset huts can you design that have approximately the same volume as this model? How do the surface areas of your new huts compare to the original model?

To explore the answers to these questions and more, check out the Performance Task and Real-Life STEM video at BigIdeasMath.com.

ANSWERS
1. The Complex Conjugates Theorem states that the polynomial has to have real coefficients. Because the term $5i$ is an imaginary number, the Complex Conjugates Theorem does not apply.
2. The turning points of the graph can be used to decide which direction the graph has been translated. For example, the point that is the local maximum on $f$ can be located on the translated graph to determine which direction the graph has been translated.
3.1 Graphing Polynomial Functions (pp. 111–118)

Graph \( f(x) = x^3 + 3x^2 - 3x - 10 \).

To graph the function, make a table of values and plot the corresponding points. Connect the points with a smooth curve and check the end behavior.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>-1</td>
<td>0</td>
<td>-5</td>
<td>-10</td>
<td>-9</td>
<td>4</td>
<td>35</td>
</tr>
</tbody>
</table>

The degree is odd and the leading coefficient is positive. So, \( f(x) \to -\infty \) as \( x \to -\infty \) and \( f(x) \to \infty \) as \( x \to \infty \).

Decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

1. \( h(x) = -x^3 + 2x^2 - 15x^2 \)
2. \( p(x) = x^3 - 5x^2 + 13x^2 + 8 \)

Graph the polynomial function.

3. \( h(x) = x^3 + 6x^2 - 5 \)
4. \( f(x) = 3x^2 - 5x^2 + 1 \)
5. \( g(x) = -x^3 + x + 2 \)

3.2 Adding, Subtracting, and Multiplying Polynomials (pp. 119–128)

a. Multiply \((x - 2)(x - 1)\) and \((x + 3)\) in a horizontal format.

\[
(x - 2)(x - 1)(x + 3) = (x^2 - 3x + 2)(x + 3)
= (x^3 - 3x^2 + 2x + 3x^2 - 9x + 6)
= x^3 - 7x + 6
\]

b. Use Pascal’s Triangle to expand \((4x + 2)^4\).

The coefficients from the fourth row of Pascal’s Triangle are 1, 4, 6, 4, and 1.

\[
(4x + 2)^4 = 1(4x)^4 + 4(4x)^3(2) + 6(4x)^2(2)^2 + 4(4x)(2)^3 + 1(2)^4
= 256x^4 + 512x^3 + 384x^2 + 128x + 16
\]

Find the sum or difference.

6. \((4x^2 - 12x^2 - 5) - (-3x^2 + 4x + 3)\)
7. \((x^4 + 3x^3 - x^2 + 6) + (2x^4 - 3x + 9)\)
8. \((3x^2 + 9x + 13) - (x^2 - 2x + 12)\)

b. Use the Binomial Theorem to expand the binomial.

12. \((m + 4)^4\)
13. \((3s + 2)^5\)
14. \((z + 1)^6\)
3.3 Dividing Polynomials (pp. 129–134)

Use synthetic division to evaluate \( f(x) = -2x^3 + 4x^2 + 8x + 10 \) when \( x = -3 \).

\[
\begin{array}{c|cccc}
-3 & -2 & 4 & 8 & 10 \\
 & & 6 & -30 & 66 \\
\hline
 & -2 & 8 & -22 & 76
\end{array}
\]

The remainder is 76. So, you can conclude from the Remainder Theorem that \( f(-3) = 76 \).

You can check this by substituting \( x = -3 \) in the original function.

Check

\[
f(-3) = -2(-3)^3 + 4(-3)^2 + 8(-3) + 10 = 54 + 36 - 24 + 10 = 76
\]

Divide using polynomial long division or synthetic division.

15. \((x^3 + x^2 + 3x - 4) ÷ (x^2 + 2x + 1)\)
16. \((x^4 + 3x^3 - 4x^2 + 5x + 3) ÷ (x^2 + x + 4)\)
17. \((x^3 - x^2 - 7) ÷ (x + 4)\)
18. Use synthetic division to evaluate \( f(x) = 4x^3 + 2x^2 - 4 \) when \( x = 5 \).

3.4 Factoring Polynomials (pp. 135–142)

a. Factor \( x^4 + 8x \) completely.

\[
x^4 + 8x = x(x^3 + 8) = x(x + 2)^3 \text{ Factor common monomial.}
\]

Write \( x^4 + 8 \) as \( a^3 + b^3 \).

\[
x(x + 2)(x^2 - 2x + 4) \text{ Sum of Two Cubes Pattern}
\]

b. Determine whether \( x + 4 \) is a factor of \( f(x) = x^4 + 4x^3 + 2x + 8 \).

Find \( f(-4) \) by synthetic division.

\[
\begin{array}{c|cccc}
-4 & 1 & 4 & 0 & 0 & 2 & 8 \\
 & & -4 & 0 & 0 & 0 & -8 \\
\hline
 & 1 & 0 & 0 & 0 & 2 & 0
\end{array}
\]

Because \( f(-4) = 0 \), the binomial \( x + 4 \) is a factor of \( f(x) = x^4 + 4x^3 + 2x + 8 \).

Factor the polynomial completely.

19. \( 64x^5 - 8 \)
20. \( 2b^5 - 12b^3 + 10c \)
21. \( 2a^5 - 7a^3 - 8a + 28 \)
22. Show that \( x + 2 \) is a factor of \( f(x) = x^4 + 2x^3 - 27x - 54 \). Then factor \( f(x) \) completely.
3.5 Solving Polynomial Equations (pp. 145–152)

a. Find all real solutions of \(x^2 + x^2 - 8x - 12 = 0\).

Step 1 List the possible rational solutions. The leading coefficient of the polynomial \(f(x) = x^2 + x^2 - 8x - 12\) is 1, and the constant term is \(-12\). So, the possible rational solutions of \(f(x) = 0\) are

\[
\begin{align*}
x & = \pm \frac{1}{1}, \pm \frac{2}{1}, \pm \frac{3}{1}, \pm \frac{4}{1}, \pm \frac{6}{1}, \pm \frac{12}{1}.
\end{align*}
\]

Step 2 Test possible solutions using synthetic division until a solution is found.

\[
\begin{array}{c|cccc}
2 & 1 & 1 & -8 & -12 \\
 & & 2 & 6 & -4 \\
\hline
 & 1 & 3 & -2 & -16 \\
 & & 2 & 6 & -4 \\
\end{array}
\]

So, \(x = -2\) is a solution. The remaining solution is found by factoring.

\[
f(2) = 0, \text{ so } x - 2 \text{ is not a factor of } f(x), \quad f(-2) = 0, \text{ so } x + 2 \text{ is a factor of } f(x).
\]

Step 3 Factor completely using the result of synthetic division.

\[
(x + 2)(x^2 - x - 6) = 0 \quad \text{Write as a product of factors.}
\]

\[
(x + 2)(x + 2)(x - 3) = 0 \quad \text{Factor the trinomial.}
\]

So, the solutions are \(x = -2\) and \(x = 3\).

b. Write a polynomial function \(f\) of least degree that has rational coefficients, a leading coefficient of 1, and the zeros \(-4\) and \(1 + \sqrt{2}\).

By the Irrational Conjugates Theorem, \(1 - \sqrt{2}\) must also be a zero of \(f\).

\[
f(x) = (x + 4)(x - (1 + \sqrt{2}))(x - (1 - \sqrt{2})) \quad \text{Write } f(x) \text{ in factored form.}
\]

\[
= (x + 4)((x - 1) - \sqrt{2})(x - 1 + \sqrt{2}) \quad \text{Regroup terms.}
\]

\[
= (x + 4)((x - 1)^2 - 2) \quad \text{Multiply.}
\]

\[
= (x + 4)(x^2 - 2x + 1 - 2) \quad \text{Expand binomial.}
\]

\[
= (x + 4)x^2 - 2x - 1 \quad \text{Simplify.}
\]

\[
= x^3 - 2x^2 - x + 4x^2 - 8x - 4 \quad \text{Multiply.}
\]

\[
x^3 + 2x^2 - 9x - 4 \quad \text{Combine like terms.}
\]

Find all real solutions of the equation.

23. \(x^3 + 3x^2 - 10x - 24 = 0\)

Write a polynomial function \(f\) of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros.

25. 1, 2, \(-\sqrt{3}\)  
26. 2, 3, \(\sqrt{5}\)  
27. \(-2, 3, + \sqrt{6}\)

28. You use 240 cubic inches of clay to make a sculpture shaped as a rectangular prism. The width is 4 inches less than the length and the height is 2 inches more than three times the length. What are the dimensions of the sculpture? Justify your answer.
3.6  The Fundamental Theorem of Algebra  (pp. 153–160)

Find all zeros of \( f(x) = x^4 + 2x^3 + 6x^2 + 18x - 27 \).

**Step 1** Find the rational zeros of \( f \). Because \( f \) is a polynomial function of degree 4, it has four zeros. The possible rational zeros are \( \pm 1, \pm 3, \pm 9 \), and \( \pm 27 \). Using synthetic division, you can determine that 1 is a zero and \(-3\) is also a zero.

**Step 2** Write \( f(x) \) in factored form. Dividing \( f(x) \) by its known factors \( x - 1 \) and \( x + 3 \) gives a quotient of \( x^2 + 9 \). So,

\[
f(x) = (x - 1)(x + 3)(x^2 + 9).
\]

**Step 3** Find the complex zeros of \( f \). Solving \( x^2 + 9 = 0 \), you get \( x = \pm 3i \). This means \( x^2 + 9 = (x + 3i)(x - 3i) \).

\[
f(x) = (x - 1)(x + 3)(x + 3i)(x - 3i).
\]

From the factorization, there are four zeros. The zeros of \( f \) are \( 1, -3, -3i, \) and \( 3i \).

Write a polynomial function \( f \) of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros.

29. \( 3, 1 + 2i \)  
30. \( -1, 2, 4i \)  
31. \( -5, -4, 1 - \sqrt{3}i \)

Determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for the function.

32. \( f(x) = x^4 - 10x + 8 \)  
33. \( f(x) = -6x^4 - x^3 + 3x^2 + 2x + 18 \)

3.7  Transformations of Polynomial Functions  (pp. 161–166)

Describe the transformation of \( f(x) = x^2 \) represented by \( g(x) = (x - 6)^2 - 2 \). Then graph each function.

Notice that the function is of the form \( g(x) = (x - h)^2 + k \). Rewrite the function to identify \( h \) and \( k \).

\[
g(x) = (x - 6)^2 + (-2)
\]

Because \( h = 6 \) and \( k = -2 \), the graph of \( g \) is a translation 6 units right and 2 units down of the graph of \( f \).

Describe the transformation of \( f \) represented by \( g \). Then graph each function.

34. \( f(x) = x^4, g(x) = (-x)^4 + 2 \)  
35. \( f(x) = x^4, g(x) = -(x + 9)^4 \)

Write a rule for \( g \).

36. Let the graph of \( g \) be a horizontal stretch by a factor of 4, followed by a translation 3 units right and 5 units down of the graph of \( f(x) = x^4 + 3x \).
37. Let the graph of \( g \) be a translation 5 units up, followed by a reflection in the y-axis of the graph of \( f(x) = x^4 - 2x^3 - 12 \).
ANSWERS

38. The $x$-intercepts of the graph are $x = -1.68$. The function has a local maximum at $(-1, -2)$ and a local minimum at $(1, 3)$. The function is increasing when $-1 < x < 0$ and decreasing when $x < -2$ and $x > 0$.

39. The $x$-intercepts of the graph are $x = 0.25$ and $x = 1.34$. The function has a local maximum at $(-1.13, 7.06)$ and local minimums at $(-2, 6)$ and $(0.88, 3.17)$. The function is increasing when $-2 < x < -1.13$ and $x > 0.88$ and is decreasing when $x < -2$ and $-1.13 < x < 0.88$.

40. odd
41. even
42. neither
43. $f(x) = \frac{1}{10}(x + 4)(x - 4)(x - 2)$
44. $f(x) = 2x^3 - 7x^2 - 6x$

### 3.8 Analyzing Graphs of Polynomial Functions (pp. 167–174)

Graph the function $f(x) = x(x + 2)(x - 2)$. Then estimate the points where the local maximums and local minimums occur.

**Step 1** Plot the $x$-intercepts. Because $-2, 0, 2$ are zeros of $f$, plot $(-2, 0), (0, 0), (2, 0)$.

**Step 2** Plot points between and beyond the $x$-intercepts.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-15</td>
</tr>
<tr>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-3</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

**Step 3** Determine end behavior. Because $f(x)$ has three factors of the form $x - k$ and a constant factor of 1, $f(x)$ is a cubic function with a positive leading coefficient. So $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

**Step 4** Draw the graph so it passes through the plotted points and has the appropriate end behavior.

- The function has a local maximum at $(1.15, 3.08)$ and a local minimum at $(-1.15, -3.08)$.

Graph the function. Identify the $x$-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which the function is increasing or decreasing.

38. $f(x) = -2x^3 - 3x^2 - 1$  \hspace{1cm} 39. $f(x) = x^4 + 3x^3 - x^2 - 8x + 2$

Determine whether the function is even, odd, or neither.

40. $f(x) = 2x^3 + 3x$  \hspace{1cm} 41. $g(x) = 3x^2 - 7$  \hspace{1cm} 42. $h(x) = x^6 + 3x^4$

### 3.9 Modeling with Polynomial Functions (pp. 175–180)

Write the cubic function whose graph is shown.

**Step 1** Use the three $x$-intercepts to write the function in factored form.

$f(x) = a(x + 3)(x + 1)(x - 2)$

**Step 2** Find the value of $a$ by substituting the coordinates of the point $(0, -12)$.

$-12 = a(0 + 3)(0 + 1)(0 - 2)$

$a = 2$

- The function is $f(x) = 2(x + 3)(x + 1)(x - 2)$.

43. Write a cubic function whose graph passes through the points $(-4, 0), (4, 0), (0, 6)$, and $(2, 0)$.

44. Use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function.

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>-11</td>
<td>-24</td>
<td>-27</td>
<td>-8</td>
<td>45</td>
<td>144</td>
<td>301</td>
</tr>
</tbody>
</table>
Write a polynomial function \( f \) of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros.

1. \( 3, 1 - \sqrt{2} \)
2. \(-2, 4, 3i\)

Find the product or quotient.

3. \((x^3 - 4x^2 - 7x + 5)(x^2 - x - 1)\)
4. \((3x^4 - 2x^2 - x + 1)(x^2 - 2x + 1)\)
5. \((2x^3 + 3x^2 + 5x - 1)(x + 2)\)
6. \((2x + 3)^3\)

7. The graphs of \( f(x) = x^4 \) and \( g(x) = (x - 3)^4 \) are shown.
   a. How many zeros does each function have? Explain.
   b. Describe the transformation of \( f \) represented by \( g \).
   c. Determine the intervals for which the function \( g \) is increasing or decreasing.

8. The volume \( V \) (in cubic feet) of an aquarium is modeled by the polynomial function \( V(x) = x^3 + 2x^2 - 13x + 10 \), where \( x \) is the length of the tank.
   a. Explain how you know \( x = 4 \) is not a possible rational zero.
   b. Show that \( x = 1 \) is a zero of \( V(x) \). Then factor \( V(x) \) completely.
   c. Find the dimensions of the aquarium shown.

9. One special product pattern is \((a - b)^2 = a^2 - 2ab + b^2\). Using Pascal’s Triangle to expand \((a - b)^2\) gives \(1a^2 + 2a(-b) + 1(-b)^2\). Are the two expressions equivalent? Explain.

10. Can you use the synthetic division procedure that you learned in this chapter to divide any two polynomials? Explain.

11. Let \( T \) be the number (in thousands) of new truck sales. Let \( C \) be the number (in thousands) of new car sales. During a 10-year period, \( T \) and \( C \) can be modeled by the following equations where \( t \) is time (in years).
    \[
    T = 23t^4 - 330t^2 + 3500t + 9000
    
    C = 14t^3 - 240t^2 + 5900t + 8900
    
    a. Find a new model \( S \) for the total number of new vehicle sales.
    b. Is the function \( S \) even, odd, or neither? Explain your reasoning.

12. Your friend has started a golf caddy business. The table shows the profits \( p \) (in dollars) of the business in the first 5 months. Use finite differences to find a polynomial model for the data. Then use the model to predict the profit after 7 months.

<table>
<thead>
<tr>
<th>Month, ( t )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profit, ( p )</td>
<td>4</td>
<td>2</td>
<td>6</td>
<td>22</td>
<td>56</td>
</tr>
</tbody>
</table>

**ANSWERS**

1. \( f(x) = x^3 - 5x^2 + 5x + 3 \)
2. \( f(x) = x^4 - 2x^3 + x^2 - 18x - 72 \)
3. \( x^8 - 7x^6 + 5x^4 - 4x^2 + 28x - 20 \)
4. \( 3x^2 + 4x + 5 + \frac{5x - 6}{x^2 - 2x + 1} \)
5. \( 2x^2 - 7x + 19 = \frac{39}{x + 2} \)
6. \( 8x^3 + 36x^2 + 54x + 27 \)
7. a. Each function has one zero repeated four times; The zero for \( f \) is 0 and the zero for \( g \) is 3.
   b. The graph of \( g \) is a translation 3 units right of the graph of \( f \).
   c. \( g \) is increasing when \( x > 3 \) and decreasing when \( x < 3 \).

8. a. The Rational Root Theorem states that all possible rational zeros of \( V(x) \) will be of the form factors of 10
   Because 4 is not a factor of 10, 4 is not a possible rational zero of \( V(x) \).
   b. \[
   1 \quad 1 \\
   2 \quad 10 \\
   3 \quad 10 \\
   3 \quad 10 \\
   1 \\
   0
   \]
   \( V(x) = (x - 1)(x + 5)(x - 2) \)
   c. The aquarium is 1.31 feet by 7.31 feet by 0.31 foot.

9. Yes; Because \((−b)^2 = b^2\), the expression \(a^2 - 2ab + b^2\) simplifies to \(a^2 - 2ab + b^2\).

10. No; The synthetic division procedure in this chapter can only be used when the divisor is a binomial of the form \( x - k \).

11. a. \( S = 37t^4 - 660t^3 + 5900t^2 - 13,400t + 17,900 \)
   b. Neither; The function is not odd because \( S(-x) \neq -S(x) \), and the function is not even because \( S(-x) \neq S(x) \)

12. \( p = t^3 - 3t^2 + 6; 202 \text{ dollars} \)
Cumulative Assessment

1. The synthetic division represents \( f(x) \div (x - 3) \). Choose a value for \( m \) so that \( x - 3 \) is a factor of \( f(x) \). Justify your answer.

\[
\begin{array}{cccccc}
3 & 1 & -3 & m & 3 & -3 \\
 & & 3 & 0 & 3 & 3 \\
1 & 0 & -2 & 2 & -1 & 1 \\
\end{array}
\]

2. Analyze the graph of the polynomial function to determine the sign of the leading coefficient, the degree of the function, and the number of real zeros. Explain.

3. About 52,300 people live in a 3-kilometer radius of a city’s center. Ten years ago, the population density in this region was about 1750 people per square kilometer. Which statement is not true?

- A. The area of the region is \( 9\pi \) square kilometers.
- B. The current population density is about 1850 people per square kilometer.
- C. Ten years ago, there were about 49,480 people living in this region.
- D. The current population density is less than it was 10 years ago.

4. A parabola passes through the point shown in the graph. The equation of the axis of symmetry is \( x = -a \). Which of the given points could lie on the parabola? If the axis of symmetry was \( x = a \), then which points could lie on the parabola? Explain your reasoning.
5. Select values for the function to model each transformation of the graph of 
\[ f(x) = x. \]
\[ g(x) = \boxed{(x - 2)} + \boxed{3}. \]

a. The graph is a translation 2 units up and 3 units left.
b. The graph is a translation 2 units right and 3 units down.
c. The graph is a vertical stretch by a factor of 2, followed by a translation 2 units up.
d. The graph is a translation 3 units right and a vertical shrink by a factor of \(\frac{1}{2}\), followed by a translation 4 units down.

6. Which description represents the solid produced by rotating the figure around the given axis?

- A cone with a height of 6 and a radius of 8
- B cone with a height of 8 and a radius of 6
- C pyramid with a height of 6 and a square base whose edge length is 8
- D pyramid with a height of 8 and a square base whose edge length is 6

7. Classify each function as even, odd, or neither. Justify your answer.
   a. \( f(x) = 3x^3 \)
   b. \( f(x) = 4x^3 + 8x \)
   c. \( f(x) = 3x^3 + 12x^2 + 1 \)
   d. \( f(x) = 2x^4 \)
   e. \( f(x) = x^{11} - x^7 \)
   f. \( f(x) = 2x^8 + 4x^4 + x^2 - 5 \)

8. The volume of the rectangular prism shown is given by \( V = 2x^3 + 7x^2 - 18x - 63 \). Which polynomial represents the area of the base of the prism?
   a. \( 2x^2 + x - 21 \)
   b. \( 2x^2 + 21 - x \)
   c. \( 13x + 21 + 2x^2 \)
   d. \( 2x^2 - 21 - 13x \)

9. The number \( R \) (in tens of thousands) of retirees receiving Social Security benefits is represented by the function

\[ R = 0.286t^3 - 4.68t^2 + 8.8t + 403, \quad 0 \leq t \leq 10 \]

where \( t \) represents the number of years since 2000. Identify any turning points on the given interval. What does a turning point represent in this situation?
Integrated Mathematics III Ancillaries

Assessment Book .......................................................... A2
Resources by Chapter .................................................. A5
Student Journal ............................................................ A8
Differentiating the Lesson ............................................. A10
**Assessment Book**

The **Assessment Book** contains formative and summative assessment options, providing teachers with the ability to assess students on the same content in a variety of ways. It is available in print and online in an editable format.

**Prerequisite Skills Test with Item Analysis**

The Prerequisite Skills Test checks students’ understanding of previously learned mathematical skills they will need to be successful in their math class. You can use the Item Analysis to diagnose topics your students need to review to prepare them for the school year.

**Pre-Course Test with Item Analysis/Post-Course Test with Item Analysis**

The Pre-Course Test and Post-Course Test cover key concepts that students will learn in their math class. You can gauge how much your students learned throughout the year by comparing their Pre-Test score against their Post-Test score. These tests also can be given as practice for state assessments.
Chapter Quiz
The Quiz provides ongoing assessment of student understanding. You can use this as a gradable quiz or as practice.

Chapter Tests
The Chapter Tests provide assessment of student understanding of key concepts taught in the chapter. There are two tests for each chapter. You can use these as gradable tests or as practice for your students to master an upcoming test.

Alternative Assessment
Each Alternative Assessment includes at least one multi-step problem that combines a variety of concepts from the chapter. Students are asked to explain their solutions, write about the mathematics, or compare and analyze different situations. You can use this as an alternative to traditional tests. It can be graded, assigned as a project, or given as practice.
Performance Task

The Performance Task presents an assessment in a real-world situation. Every chapter has a task that allows students to work with multiple standards and apply their knowledge to realistic scenarios. You can use this task as an in-class project, a take-home assignment or as a graded assessment.

Chapter 3 Performance Task (continued)

For the Birds—Wildlife Management

How does the presence of humans affect the population of sparrows in a park? Do more humans mean fewer sparrows? Or does the presence of humans increase the number of sparrows up to a point? Are there a minimum number of sparrows that can be found in a park, regardless of how many humans are there? What can a mathematical model tell you?

In 1997, researchers set out to answer these questions. They observed the sparrow population and the numbers of pedestrians in wooded parks. Their approximate data can be seen below.

1. Make a scatter plot of the data. Explain your domain and range.
2. Does the data represent a function? Why or why not? What might explain the duplicate entries?
3. What trends do you see in the data?
4. Using the regression feature of your calculator, find three models for this data along with the corresponding coefficients of determination.
   a. a linear model
   b. a quadratic model
   c. a cubic model

<table>
<thead>
<tr>
<th>Number of pedestrians per hectare</th>
<th>Number of sparrows per hectare</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>80</td>
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<tr>
<td>5</td>
<td>75</td>
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<td>6</td>
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<td>12</td>
<td>140</td>
</tr>
<tr>
<td>13</td>
<td>179</td>
</tr>
</tbody>
</table>

Quarterly Standards Based Test

The Quarterly Standards Based Test provides students practice answering questions in standardized test format. The assessments are cumulative and cover material from multiple chapters of the textbook. The questions represent problem types and reasoning patterns frequently found on standardized tests. You can give this test to your students as a cumulative assessment for the quarter or as practice for state assessment.

Quarterly Standards Based Test

Chapters 1–3

1. Which function does not belong with the other three?
   a. \( f(x) = x^2 - 3 \)
   b. \( f(x) = x^2 + 5 \)
   c. \( f(x) = (x - 4)^2 \)
   d. \( f(x) = x - 3 \)

2. If a function has a domain of all real numbers and a range of \( y \geq 3 \), which of the following are possible functions? Choose all that apply.
   a. \( g(x) = x^2 + 3 \)
   b. \( g(x) = -2x + 8 \)
   c. \( g(x) = \sqrt{x} - 2 \)
   d. \( g(x) = x + 3 \)

3. Let \( f(x) = 2x - 5y^2 - 4 \). Which of the following functions is a reflection in the \( x \)-axis of the graph of \( f \)?
   a. \( g(x) = -2x + 5y^2 - 4 \)
   b. \( g(x) = -2x + 5y^2 + 4 \)
   c. \( g(x) = 2x + 5y^2 - 4 \)
   d. \( g(x) = 2x + 5y^2 + 4 \)

4. Write a function \( g \) whose graph represents the indicated transformation of each graph of \( f \). Use a graphing calculator to check your answers.
   a. \( f(x) = x^3 - 2 \); reflection in the \( x \)-axis
   b. \( f(x) = -x^3 - 2 \); vertical shrink by a factor of \( \frac{1}{2} \)
   c. \( f(x) = \sqrt{x} - 4 \); horizontal shrink by a factor of \( \frac{1}{4} \)
   d. \( f(x) = 4x - 3 \); translation 3 units left
Resources by Chapter

The Resources by Chapter ancillary includes a number of supplemental resources for every chapter in the program. It is available in print and online in an editable format.

Family Communication Letters (English and Spanish)

The Family Communication Letters provide a way to quickly communicate to family members how they can help their student with the material of the chapter. You can send this home with your students to help make the mathematics less intimidating and provide suggestions for families to help their students see mathematical concepts in common activities.

Dear Family,

There are many patterns in the world that can be explained using math. Once a pattern is determined, the information can be used to make predictions and simplify problems. One of the most commonly used patterns in mathematics is Pascal's Triangle, named after the French mathematician Blaise Pascal.

Pascal’s Triangle, named after the French mathematician Blaise Pascal.

To find the next row of Pascal’s Triangle, the first and last numbers will be 1. Then each number between each 1 is the sum of the two numbers above it.

Work together to copy the first 5 rows of Pascal’s Triangle onto a sheet of paper, and then find the next 10 rows of the pattern. Use the first 15 rows of Pascal’s Triangle to answer the following questions:

• What pattern do you notice within the triangle?
• What pattern do you notice when you find the sum of each row?
• What is the relationship between Pascal’s Triangle and the powers of 2?

You can use the Internet to verify the rows of the Triangle and the answers to the questions above.

In this chapter, your student will work with polynomial functions. To find the binomial expansion of \((x + y)^n\), your student can use Pascal’s Triangle to determine the coefficients of the variables.

There is a lot of interesting information about Pascal’s Triangle, including its history and applications in the mathematical and real world, such as in probability and statistics.

Have fun looking for patterns in your daily life!

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3.1 Start Thinking

Use a graphing calculator to graph the functions \( f(x) = x^3 \) and \( g(x) = x^4 \). Compare and contrast the graphs of the two functions.

Explain why the graph of \( g(x) = x^4 \) is always positive. Are there any points the two graphs have in common? If so, what are they?

3.1 Warm Up

Evaluate the function for the given value of \( x \).

1. \( f(x) = 7x - 6; x = -2 \)
2. \( g(x) = x^2 + 3; x = 6 \)
3. \( f(x) = -3x + 4; x = -2 \)
4. \( g(x) = x^2 - 6x; x = -4 \)
5. \( f(x) = 1.7x - 7; x = 16 \)
6. \( h(x) = 8.49x; x = -4 \)

3.1 Cumulative Review Warm Up

Determine whether the given characteristics describe a parabola that opens up or down.

1. Focus: \((0, -5)\)
   Directrix: \(y = 5\)
2. Focus: \((0, 5)\)
   Directrix: \(y = -5\)
3. Focus: \((0, -1)\)
   Directrix: \(y = 1\)
4. Focus: \((0, 1)\)
   Directrix: \(y = -1\)

Practice

The Practice exercises provide additional practice on the key concepts taught in the lesson. There are two levels of practice provided for each lesson: A (basic) and B (average). You can assign these exercises for students that need the extra work or as a gradable assignment.

3.1 Practice A

In Exercises 1–4, decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

1. \( f(x) = 3x^4 - 5x^3 + 2x^2 - x - 4 \)
2. \( h(x) = 5x^4 - 3x^3 + x^2 - x + 1 \)
3. \( g(x) = 3x^3 - 4x^2 + 5x - 3 \)
4. \( f(x) = x^4 - 2x^3 + 3x - 1 \)

In Exercises 5–7, evaluate the function for the given value of \( x \).

1. \( f(x) = 3x^4 - 5x^3 + 2x^2 - x - 4; x = 1 \)
2. \( g(x) = 3x^3 - 4x^2 + 5x - 3; x = -1 \)
3. \( h(x) = x^4 - 2x^3 + 3x - 1; x = 0 \)

In Exercises 8 and 9, describe the end behavior of the graph of the function.

1. \( g(x) = 3x^4 - 5x^3 + 2x^2 - x - 4 \)
2. \( h(x) = 3x^3 - 4x^2 + 5x - 3 \)
3. \( f(x) = x^4 - 2x^3 + 3x - 1 \)

In Exercises 10–13, graph the polynomial function.

10. \( g(x) = x^4 - 2 \)
11. \( h(x) = x^2 - 2x + 3 \)
12. \( f(x) = 2x^2 - 3x \)
13. \( x^4 - 2x^3 + 3x - 1 \)

14. Suppose \( f(x) \to -\infty \) as \( x \to -\infty \) and \( f(x) \to \infty \) as \( x \to \infty \). Describe the degree and leading coefficient of the function.
In Exercises 1–12, simplify the expression.

Name ____________________________  Date __________

To input data, press [STAT] and then [EDIT]. Input x-values starting with 0 and y-values. To calculate the Quartic Regression, press [STAT] and then [CALC] and select 7:QuartReg. Press [VARS] and select Y-VARS, then select 1:Function and 1:Y1. 

The equation \( y = 0.00002x^4 + 0.0225x^3 + 0.694x^2 + 14.46x + 256.6 \)

appears on the screen. Press [ENTER].

The quaratic regression equation for the data given in the table.

Each Enrichment and Extension extends the lesson and provides a challenging application of the key concepts. These rigorous exercises can be assigned to challenge your students to use higher order thinking skills or to build on a concept via an extension of the lesson.

Each Puzzle Time provides additional practice in a fun format in which students use their mathematical knowledge to solve a riddle. This format allows students to self-check their work. Your students can learn the lesson concepts by finding the answers to silly jokes and riddles.

The Cumulative Review includes exercises covering concepts and skills from the current chapter and previous chapters. Students can work on their mastery of previously learned material.
Student Journal

The Student Journal serves as a valuable component where students may work extra problems, take notes about new concepts and classroom lessons, and internalize new concepts by expressing their findings in their own words. Available in English and Spanish

Maintaining Mathematical Proficiency

The Maintaining Mathematical Proficiency corresponds to the Pupil Edition Chapter Opener. Your students have the opportunity to practice prior skills necessary to move forward.

1. \(-9x - 9\)
2. \(2x - 5 + 5 - x\)
3. \(3 - (2x - 5) \pm x\)
4. \(5x - (2x - 5) + 11\)
5. \(-3x + 7(1) - 2(4)\)
6. \(5 - 6n + 5n + 6n^2\)

Find the volume or surface area of the solid.

7. Volume of a right cylinder with radius 5 feet and height 15 feet
8. Surface area of a rectangular prism with length 10 meters, width 20 meters, and height 4 meters
9. Volume of a cube with side length 2.5 millimeters
10. Surface area of a sphere with radius 1 foot
11. For what radius length can the value of the volume of a sphere equal the value of the surface area?

Exploration Journal

The Exploration pages correspond to the Explorations and accompanying exercises in the Pupil Edition. Your students can use the room given on these pages to show their work and record their answers.

3.1 Graphing Polynomial Functions

For use with Exploration 3.1

Essential Question. What are some common characteristics of the graphs of cubic and quartic polynomial functions?

EXPLORATION: Identifying Graphs of Polynomial Functions

Go to BigIdeasMath.com for an interactive tool that investigates the exploration.

Work with a partner. Match each polynomial function with its graph. Explain your reasoning. Use a graphing calculator to verify your answers.

\[ f(x) = x^3 \]
\[ f(x) = x^3 - x \]
\[ f(x) = x^3 + 3x \]
\[ f(x) = x^3 + 1 \]

A. \[ f(x) = x^3 \]
B. \[ f(x) = x^3 - x \]
C. \[ f(x) = x^3 + 3x \]
D. \[ f(x) = x^3 + 1 \]
E. \[ f(x) = x^3 - x \]
F. \[ f(x) = x^3 + 1 \]
Notetaking with Vocabulary

This student-friendly notetaking component is designed to be a reference for key vocabulary, properties, and core concepts from the lesson. Students can write the definitions of the vocabulary terms in their own words and take notes about the core concepts.

Extra Practice

Each section of the Pupil Edition has an Extra Practice in the Student Journal with room for students to show their work.
Differentiating the Lesson

The **Differentiating the Lesson** online ancillary provides complete teaching notes and worksheets that address the needs of diverse learners. Lessons engage students in activities that often incorporate visual and kinesthetic learning. Some lessons present an alternative approach to teaching the content, while other lessons extend the concepts of the text in a challenging way for advanced students. Each chapter also begins with an overview of the differentiated lessons in the chapter and describes the students who would most benefit from the approach used in each lesson.

![Lesson Preparation](image)

**Materials:** Resource Sheet 3: Dividing Polynomials, scientific calculators (as needed)

**Reference:** Pre-assess students’ fluency with multiplication facts as well as their ability to multiply polynomials. Ensure that students are familiar with the box method for multiplying polynomials, as outlined in Differentiating the Lesson—Chapter 7, Lesson 2. Make copies of Resource Sheet 3 (one per student).

**Classroom Management:** Allow students time to work through the problems at their own pace. Consider an alternative activity for students who are able to divide quickly and accurately. If students are ready, allow them to move onto Resource Sheet 3, or to create problems for each other.

**Lesson Procedure**

Display the expression \((2x^2 + 5x - 1)/(x + 4)\) and have students use the box method to multiply the polynomials. \([3231 71 9 4 \text{xxx}++−] \[\text{The box method is shown below.}\]

\[
\begin{array}{c|cc}
 & 3x^2 & 5x \\
\hline
2x^2 & 3x^2 & 5x^2 \\
5x & 5x & -1 \\
\end{array}
\]

Discuss how the problem could be rewritten using division. Have students generate ideas for how to do this. \(\text{Sample response: } 2x^2 + 5x - 1 = \frac{2x^2 + 5x - 1}{x + 4}\)

**Note:** There is more than one way to rewrite the problem. If students have difficulty with this, consider reviewing how a multiplication problem with integers could be rewritten as a division problem. Focus on the terminology: dividend, divisor, and quotient and update to the problem.

Display the problem \(\frac{2x^2 + 5x - 1}{x + 4}\) and explain that students are going to use the box method in reverse to divide the polynomials. Have students create appropriately sized boxes and fill in the factors they have along one edge. Have students discuss how they might fill in the first cell of their boxes, considering the factor they know and the leading term. (The leading term is \(2x^2\), and \(x^2\) would have to be multiplied by 2x to arrive at \(2x^2\).) The first term in the second factor must be \(2x\).
Credits

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Chapter 3
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