Risk Management: Principles and Applications
Risk Management: Principles and Applications

Course Introduction and Overview

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1 Course Objectives

Welcome to this course on *Risk Management: Principles and Applications*. This course has four main aims, to:

- illustrate the main types of risk
- present the most important ideas and methods used in the analysis of portfolios of financial securities, including stocks and bonds
- explain how rational investors can use financial derivatives (mainly, futures and options) in order to alter the risk of their investment position
- illustrate some more specialised risk management techniques, such as Value at Risk and Credit Risk.

The emphasis throughout is on the general principles behind the investment decisions, rather than on case studies or anecdotal evidence. Thus, you will study, for instance,

- the main features of portfolios, which include stocks and bonds,
- how to calculate their risk, and
- how investors can combine their holdings of different securities to reduce their overall risk without sacrificing return.

Similarly, when you deal with futures and options, you will explore how these instruments can be used to manage risk and to expand the opportunity set of investors.

All firms face risks, although the types of risks they face and their extent differ. Consequently, management requires a strategy for dealing with risk and appropriate techniques for implementing the strategy. It may be tempting to think simply that risk is undesirable and that strategies are chosen to minimise risk, but that is not the assumption that underlies risk management principles because, in general, low levels of risk imply low levels of expected profits (expected returns).

Instead, we assume that firms choose some level of risk that gives them a desirable combination of risk and return, a desirable risk-return ratio. Some firms’ managers choose a high-risk strategy in the belief that high expected profits are associated with high risk; others choose a conservative (low-risk, low-expected-returns) strategy. In order to achieve their desired risk-return combination, management needs to be able to calculate risk, to value it, and to change the combination of risk and expected return by buying and selling assets and liabilities. The assets and liabilities in firms’ portfolios include those that take the form of derivatives contracts (such as options), which powerfully facilitate strategies that reduce risk (‘hedging’) or increase it (‘speculation’).

This course focuses on the concepts and techniques that managers use to achieve desired risk-return combinations. The course gives particular
attention to the techniques and concepts associated with derivatives. That includes hedging techniques using options and futures contracts.

Many of the principles in risk management were developed by financial firms, such as banks and investment fund managers, but they are also applicable to non-financial firms in commerce, manufacturing or other sectors. In this course we focus on the management of risk by non-financial firms in two ways. One is by showing how the principles used by financial firms are also useful for non-financial firms’ risk management. Another is by showing how risk-management principles used by banks and investment funds affect their loans to, or investment in, non-financial firms so that those non-financial firms have to take them into account in their own management decisions.

2 The Course Author


Professor Scaramozzino has taught *Risk Management* for the on-campus MSc in Finance and Financial Law in London and has contributed to several off-campus CeFiMS courses, including *Mathematics and Statistics for Economists*, *Portfolio Analysis and Derivatives*, *Quantitative Methods for Financial Management*, *Managerial Economics* and *Derivatives*.

Dr Jonathan Simms provided revisions and additional material. Dr Simms is a tutor for CeFiMS, and has taught at University of Manchester, University of Durham and University of London. He has contributed to development of various CeFiMS courses including *Econometric Principles and Data Analysis*, *Econometric Analysis and Applications*, *Financial Econometrics*, *Introduction to Valuation*, *Advanced Topics in Valuation*, *Public Financial Management: Reporting and Audit*, *Banking Strategy*, and *Introduction to Law and to Finance*. 
3. The Course Structure

Unit 1 Introduction to Risk Management
1.1 Introduction to Portfolio Analysis
1.2 Risks Faced by Financial and Non-financial Institutions
1.3 Financial Securities and Financial Markets
1.4 The Mean-Variance Approach
1.5 The Opportunity Set under Risk - Efficient Portfolios
1.6 Short Sales and Riskless Lending and Borrowing
1.7 How to Compute the Efficient Set
1.8 Conclusions

Unit 2 Portfolio Analysis
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Unit 3 Management of Bond Portfolios
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3.5 Duration
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3.7 Passive Bond Portfolio Management – Matching, Immunisation, Indexation
3.8 Active Bond Portfolio Management – Index Models
3.9 Active Bond Portfolio Management – Swaps
3.10 Conclusions

Unit 4 Futures Markets
4.1 Introduction
4.2 Description of Financial Futures
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4.5 Examples of Using Futures
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4.8 Conclusions

Unit 5 Options Markets
5.1 Introduction
5.2 Features of Options Contracts
5.3 Options on Stocks and Futures
5.4 Risk Exposure and Profit Potential of Options and Futures
5.5 The Put-Call Parity Formula
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5.7 Pricing of Options on Futures
5.8 Price Volatility
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Unit 6 Risk Management with Options
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Unit 7 Value at Risk
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8.2 Credit Rating Systems
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8.6 Valuation of Bonds
8.7 Forward Distribution of Changes in the Value of Bonds
8.8 Credit VaR for a Bond or Loan Portfolio
8.9 Credit VaR and Calculation of Capital Charge
8.10 Conclusions

4 Learning Outcomes

When you have completed this course, you will be able to do the following:

- outline the most important strategies of risk management
- explain how stocks and bonds can contribute to the risk and return of a financial portfolio
- discuss the key principles of diversification of financial investment
- correctly measure the risk of financial portfolios
- explain the risk profile involved in financial derivatives, such as futures and options
• discuss the importance of Value at Risk and scenario analysis
• define and use the principles of credit risk analysis.

5 Study Materials

The materials provided for this course comprise the course guide, presented in eight units of text covering different topics.

The course units illustrate the main ideas underlying risk analysis and suggest how to proceed in the study of each issue. The units are designed in the expectation that you will devote about 15-20 hours to studying each topic, including all the associated readings and the set exercises.

The study resources used to develop the topics comprise a book of Readings and three textbooks.

Textbooks

The three textbooks for this course are:


The textbook by Elton, Gruber, Brown and Goetzmann offers a very thorough and up-to-date presentation of modern portfolio analysis. One of its main attractions is that it seeks to balance the formal aspects of theory with the demands of practitioners. Although the text does not dwell on many institutional details, it discusses very clearly the relevance of the various theoretical results for the actual implementation of security analysis and portfolio management, and provides a number of useful examples. You will make extensive use of this book by Elton et al. for Units 1–3 of the course, dealing with the analysis of portfolios of stocks and bonds.

Hull’s textbook provides a clear treatment of futures and options markets. You will be required to study a number of chapters of the text, but if you develop a professional interest in derivatives, you will find it useful to study the whole book. For this course on Risk Management, you will use this book mainly for Units 4-6, which deal with futures and options.

Note that the textbook author John Hull has developed software called DerivaGem, which enables you to compute directly the prices of futures, options and other derivatives. The software and details on how to install and run DerivaGem are provided on the author’s website at: http://www-2.rotman.utoronto.ca/~hull/software/index.html.
The textbook by Crouhy, Galai and Mark presents a comprehensive analysis of the various sources of risk in financial markets. The authors are academics with long-standing professional expertise, and they successfully bridge the gap between the general principles and the practice of risk management. For this course, you will use their book for Unit 1 and then for Units 7 and 8, which cover two more specialised techniques of risk management: Value at Risk and Credit Risk.

Course Reader

We also provide you with academic articles, papers and reports, which are assigned as core readings in the Study Guide. They make up a Course Reader. You are expected to read them as an essential part of the course. We have selected articles and reports which reinforce your understanding of the material in the Study Guide and textbooks, and which also demonstrate how the methods you are studying are applicable and relevant in the evolving risk environment, including the financial crisis beginning 2007/08.

You will also see that some of the Course Units refer you to a number of references online which you may find useful in giving these topics an extra dimension.

Study Advice

When you study each unit, it is essential that you

- read the course units when you approach the topic
- study the relevant sections of the textbooks
- read the suggested articles from the Course Reader
- solve the problems as you are advised.

You must read and think about the textbook chapters and articles from the Course Reader you are asked to read at the points indicated. It is important that you understand each topic well, before moving on to the next one. The material presented in the course follows a logical sequence, and you will find it difficult to understand the later topics if you do not fully understand the previous ones.

The exercises and problems require that you answer some specific questions on risk management, and solve numerical problems. They are designed to test your understanding of the issues, and are also meant to provide useful practice in preparation for your examination. It is crucial that you take great care in thinking through the exercises, and answer the questions as carefully and thoroughly as you can before you proceed to the following reading.

The answers to the numerical problems are usually provided at the end of each unit.
6 Assessment

Your performance on each course is assessed through two written assignments and one examination. The assignments are written after weeks four and eight of the course session and the examination is written at a local examination centre in October.

The assignment questions contain fairly detailed guidance about what is required. All assignment answers are limited to 2,500 words and are marked using marking guidelines. When you receive your grade it is accompanied by comments on your paper, including advice about how you might improve, and any clarifications about matters you may not have understood. These comments are designed to help you increase your understanding of the subject and to improve your skills as you progress through your programme.

The written examinations are ‘unseen’ (you will only see the paper in the exam centre) and written by hand, over a three hour period. We advise that you practise writing exams in these conditions as part of your examination preparation, as it is not something you would normally do.

You are not allowed to take in books or notes to the exam room. This means that you need to revise thoroughly in preparation for each exam. This is especially important if you have completed the course in the early part of the year, or in a previous year.

Preparing for Assignments and Exams

There is good advice on preparing for assignments and exams and writing them in Sections 8.2 and 8.3 of Studying at a Distance by Talbot. We recommend that you follow this advice.

The examinations you will sit are designed to evaluate your knowledge and skills in the subjects you have studied: they are not designed to trick you. If you have studied the course thoroughly, you will pass the exam.

Understanding assessment questions

Examination and assignment questions are set to test different knowledge and skills. Sometimes a question will contain more than one part, each part testing a different aspect of your skills and knowledge. You need to spot the key words to know what is being asked of you. Here we categorise the types of things that are asked for in assignments and exams, and the words used. All the examples are from CeFiMS examination papers and assignment questions.

Definitions

Some questions mainly require you to show that you have learned some concepts, by setting out their precise meaning. Such questions are likely to be preliminary and be supplemented by more analytical questions. Generally ‘Pass marks’ are awarded if the answer only contains definitions. They will contain words such as:

- Describe
- Define
Examine
Distinguish between
Compare
Contrast
Write notes on
Outline
What is meant by
List

Reasoning
Other questions are designed to test your reasoning, by explaining cause and effect. Convincing explanations generally carry additional marks to basic definitions. They will include words such as:
- Interpret
- Explain
- What conditions influence
- What are the consequences of
- What are the implications of

Judgment
Others ask you to make a judgment, perhaps of a policy or of a course of action. They will include words like:
- Evaluate
- Critically examine
- Assess
- Do you agree that
- To what extent does

Calculation
Sometimes, you are asked to make a calculation, using a specified technique, where the question begins:
- Use indifference curve analysis to
- Using any economic model you know
- Calculate the standard deviation
- Test whether

It is most likely that questions that ask you to make a calculation will also ask for an application of the result, or an interpretation.

Advice
Other questions ask you to provide advice in a particular situation. This applies to law questions and to policy papers where advice is asked in relation to a policy problem. Your advice should be based on relevant law, principles, evidence of what actions are likely to be effective.
- Advise
- Provide advice on
- Explain how you would advise

Critique
In many cases the question will include the word ‘critically’. This means that you are expected to look at the question from at least two points of view, offering a critique of
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each view and your judgment. You are expected to be critical of what you have read. The
questions may begin
- Critically analyse
- Critically consider
- Critically assess
- Critically discuss the argument that

Examine by argument
Questions that begin with ‘discuss‘ are similar – they ask you to examine by argument, to
debate and give reasons for and against a variety of options, for example
- Discuss the advantages and disadvantages of
- Discuss this statement
- Discuss the view that
- Discuss the arguments and debates concerning

The grading scheme
Details of the general definitions of what is expected in order to obtain a
particular grade are shown below. Remember: examiners will take account of
the fact that examination conditions are less conducive to polished work than
the conditions in which you write your assignments. These criteria
are used in grading all assignments and examinations. Note that as the criteria
of each grade rises, it accumulates the elements of the grade below. Assign-
ments awarded better marks will therefore have become comprehensive in
both their depth of core skills and advanced skills.

70% and above: Distinction, as for the (60-69%) below plus:
- shows clear evidence of wide and relevant reading and an engagement
  with the conceptual issues
- develops a sophisticated and intelligent argument
- shows a rigorous use and a sophisticated understanding of relevant
  source materials, balancing appropriately between factual detail and
  key theoretical issues. Materials are evaluated directly and their
  assumptions and arguments challenged and/or appraised
- shows original thinking and a willingness to take risks

60-69%: Merit, as for the (50-59%) below plus:
- shows strong evidence of critical insight and critical thinking
- shows a detailed understanding of the major factual and/or theoretical
  issues and directly engages with the relevant literature on the topic
- develops a focussed and clear argument and articulates clearly and
  convincingly a sustained train of logical thought
- shows clear evidence of planning and appropriate choice of sources and
  methodology

50-59%: Pass below Merit (50% = pass mark)
- shows a reasonable understanding of the major factual and/or
  theoretical issues involved
• shows evidence of planning and selection from appropriate sources,
• demonstrates some knowledge of the literature
• the text shows, in places, examples of a clear train of thought or argument
• the text is introduced and concludes appropriately

45-49%: Marginal failure
• shows some awareness and understanding of the factual or theoretical issues, but with little development
• misunderstandings are evident
• shows some evidence of planning, although irrelevant/unrelated material or arguments are included

0-44%: Clear failure
• fails to answer the question or to develop an argument that relates to the question set
• does not engage with the relevant literature or demonstrate a knowledge of the key issues
• contains clear conceptual or factual errors or misunderstandings

[approved by Faculty Learning and Teaching Committee November 2006]

Specimen exam papers
Your final examination will be very similar to the Specimen Exam Paper included at the end of this Introduction and Overview. It will have the same structure and style and the range of question will be comparable.

CeFiMS does not provide past papers or model answers to papers. Our courses are continuously updated and past papers will not be a reliable guide to current and future examinations. The specimen exam paper is designed to be relevant to reflect the exam that will be set on the current edition of the course.

Further information
The OSC will have documentation and information on each year’s examination registration and administration process. If you still have questions, both academics and administrators are available to answer queries.

The Regulations are also available at [www.cefims.ac.uk/regulations.shtml](http://www.cefims.ac.uk/regulations.shtml), setting out the rules by which exams are governed.
The examination must be completed in **THREE** hours. Answer **THREE** questions, at least **ONE** from each section. Answer **ALL** parts of multi-part questions.

The examiners give equal weight to each question; therefore, you are advised to distribute your time approximately equally between three questions. The examiners wish to see evidence of your ability to use technical models and of your ability to critically discuss their mechanisms and application.

**Candidates may use their own electronic calculators in this examination provided they cannot store text; the make and type of calculator MUST BE STATED CLEARLY on the front of the answer book.**

**PLEASE DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM. IT MUST BE ATTACHED TO YOUR ANSWER BOOK AT THE END OF THE EXAMINATION.**
Answer THREE questions. Answer ALL parts of multi-part questions.

Section A
Answer at least ONE question from this section.

1. Clearly explain the assumptions behind the mean-variance approach to the analysis of financial portfolios. How can this approach be applied to risk management?

2. Explain what is meant by the parameter \( \text{beta} \) of a financial security. How can \( \text{beta} \) be estimated?

3. What is the difference between spot and forward rates? Use the term structure of interest rates to explain how they could be related.

4. Answer both parts of this question.
   a. Discuss the main strategies for risk hedging and speculation in futures markets.
   b. Suppose you have taken up a mortgage at a variable interest rate. How could you protect yourself from interest rate rises by using futures contracts?

Section B
Answer at least ONE question from this section.

5. Examine the risk exposure properties and the profit potential of option contracts. Illustrate how options can be combined to obtain synthetic futures contracts.

6. What is meant by delta hedging? Explain why it can be important for the risk management of equity portfolios.

7. Define and discuss the main approaches for the evaluation of the Value at Risk of a financial portfolio.

8. Explain what is meant by credit migration. Discuss one methodology to analyse the changes in the value of a portfolio of loans or bonds over a given time horizon.

[END OF EXAMINATION]
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Unit Content

Unit 1 introduces the subject of risk management. It illustrates the structure of the course and presents the main ideas and methods in the analysis of financial investment. You will be shown how to measure the expected return and the variance of a financial portfolio, and why it is important to look at the covariances between securities. Finally, you will see how to compute the portfolio possibilities set under risk, and how to modify it when short sales are allowed and when investors are able to lend or borrow at a safe rate.

Learning Outcomes

When you have completed your study of this unit and its readings, you will be able to

- discuss what is meant by risk management
- explain the main forms of risk faced by financial and non-financial institutions
- outline the nature of a financial portfolio
- define the characteristics of the mean–variance approach
- compute the expected return and the variance of the returns on a financial security
- compute the expected return and the variance of portfolios of financial securities
- explain the covariance between the returns on two financial securities and the correlation coefficient – how they are related and why they are important
- explain the opportunity set under risk, and define efficient and inefficient portfolios
- discuss how short sales can expand the opportunity set of investors
- assess how the opportunity set can be modified by the possibility of riskless lending and borrowing.

Readings for Unit 1


You are also invited to study Elton et al. Chapter 6, but this is an optional reading.
1.1 Introduction to Portfolio Analysis

The first unit of this module gives a general introduction to Risk Management, and lays the foundation for the analysis of investment that will be pursued in all later units of the course. You will look at the main forms of risk that investors have to face, both in financial and in non-financial institutions. The various types of risk will usually require different risk management strategies, and this will be the main concern of this course. In the present unit, you will look mainly at the basic principles behind the most fundamental risk management tool: the analysis of portfolios of risky assets, or portfolio analysis.

But what exactly is meant by ‘portfolio analysis’?

In general, a collection of assets (real or financial) is called a portfolio. If you look at your own personal wealth, for instance, you will notice that it is ordinarily held in the form of a portfolio of assets. You might hold cash, bank deposits, bonds, insurance policies, and might also own durable commodities, cars, houses, etc. In general, portfolios can be composed of both real and financial assets.

The key idea in portfolio analysis is that, when investors are trying to establish the value of their wealth, they must consider their assets as a whole. Thus, in principle, investors should not consider the value to themselves of any of their real assets independently of the value of their other real assets or of their financial securities.

The reason for that can be seen by considering the following highly simplified example. Suppose that an investor holds her wealth entirely in the form of long-term bonds and a house. Let us assume that the market value of houses in the economy is affected by the level of interest rates: you can assume, in particular, that an increase in interest rates reduces the market value of houses, due to the increase in the burden of mortgage repayments and a reduced demand for houses in the economy.

Suppose now that there is an increase in the rates of interest in the economy. This would lead to a fall in the market value of the house. However, this is also likely to result in a reduction in the market price of bonds (you will see this in more detail in Unit 3 of the course, on the management of bond portfolios). Thus, the possibility that the investor might realise capital gains or losses on her bonds is associated with the possibility of capital gains or losses on the market value of her house. In other words, the investment as a whole in the portfolio happens to be very sensitive to fluctuations in interest rates. The reason for this is that the value of both bonds and house prices tends to be sensitive to changes in the rates of interest, and their returns tend to vary in the same direction (when the price of bonds increases the same is usually true of house prices, and vice versa).
The previous extremely simple example shows that, when analysing a financial investment, you must always consider the portfolio of assets as a whole, rather than single assets in isolation. You cannot just consider the expected returns on each asset, but must also be concerned with how those returns vary together. The co-movements of the returns are a crucial feature of the risk of the portfolio. The focus of this unit, as well as of Units 2 and 3 of the course, is on portfolios, in which you look at collections of assets. You will study how you can measure the risk and return of a portfolio, and how you can select portfolios of bonds and stocks.

1.2 Risks Faced by Financial and Non-financial Institutions

The first step in risk management is the identification of the sources of risk by which institutions can be affected, and the analysis of how risk can affect their profits. In general, it is important to distinguish between financial and non-financial institutions, since there could be important differences in the way they relate to risk. Regarding financial institutions, these often act as intermediaries amongst investors with a different risk exposure. Their role is therefore to offer opportunities for diversification and the hedging of risks. Conversely, they may be asked to act so as to increase the risk profile faced by investors, if these investors want to speculate.

By contrast, non-financial institutions face risks in the course of their normal business activity. These could be related to uncertainties in output markets, exchange rate fluctuations, etc. Non-financial institutions may seek to limit the effects of risks on their profits.

The risks faced by financial institutions may be classified as follows:

1. **Market risk**: this is due to changes in prices and rates of interest in financial markets. Both the assets and the liabilities of financial institutions consist of financial securities. Hence, the value of their net position can be affected by changes in equity prices, interest rates, exchange rates, etc.

2. **Credit risk**: this risk consists of changes in the credit quality of the counterparties, which affect the financial institution’s position by altering the credit quality of its balance sheet.

3. **Liquidity risk**: this is the risk associated with the institution’s ability to raise the necessary funds to meet its needs for liquidity and/or to carry out the desired financial transactions.

4. **Operational risk**: this is the risk associated with the ordinary activities of the institutions, and can be associated, for instance, with a breakdown in software systems, management failures, fraud, human error, etc.

5. **Legal and regulatory risk**: this risk is associated with the possibility of changes in the regulatory framework or in the tax laws.
Sometimes it can also involve the difficulty of enforcing a financial contract or engaging in a transaction.

6 **Systemic risk**: this risk could be associated with a systemic collapse of the banking industry at regional, national or international level (‘domino effect’). This could be due to the occurrence of bank failures, whose consequences rapidly spread throughout the system and generate a collapse in the level of confidence towards the banking system as a whole.

Similarly, the risks faced by non-financial institutions can be classified as:

1 **Business risk**: this is the risk associated with the ordinary operations of the institutions: fluctuations in demand and supply, price changes, competitive pressures, etc.

2 **Operational risk**: this includes the risk associated with technical progress, such as the need to replace production processes due to obsolescence.

3 **Market risk**: this includes the risk to the firm’s profits due to changes in inflation, interest rates, fluctuations of exchange rates, etc.

4 **Credit risk**: this includes changes in the institution’s own credit rating or in the credit rating of its clients, which could affect the firm’s cost of obtaining funds for its investment projects.

In addition, both financial and non-financial institutions are increasingly facing reputation risk. The awareness of reputation risk is becoming more widespread following a number of accounting scandals involving large corporations. Also, the increasing complexity and use of structured financial products have led some to question the legality of related transactions, and whether such transactions are suitable for all financial institutions. The major question remains how can reputation risk be measured, and should institutions maintain capital in recognition of reputation risk?

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**Reading**

You can find a discussion of the typology of risk exposures for financial and non-financial institutions in Crouhy *et al.*, Chapter 1, Appendix 1.1, ‘Typology of risk exposures’, pp. 23–43. Please stop now and read this section of the textbook. You may want to read the remaining sections of the chapter, but that is not essential reading.

The next reading is a short article by Avinash Persaud, which provides useful explanation and intuition on liquidity risk. The author also considers the various liquidity risk positions of different types of financial institution. He explains what happens when different types of financial institution start to exhibit the same behaviour as occurred in the credit crunch of 2007/08. The article covers market risk, credit risk and systemic liquidity risk, risk traders and risk absorbers, and factors that contributed to the 2007/08 credit crunch. It also touches briefly on a number of themes you will study in other units in this course. You should read the whole article, but do not give too much attention to the detail of accounting rules, supervision and regulation.
Reading

Please now read the article ‘Regulation, valuation and systemic liquidity’ by Avinash Persaud.

Be sure your notes cover the main issues cited above.

1.3 Financial Securities and Financial Markets

The main instruments traded in financial markets are known as financial securities. The various types of financial securities involve different risks, and play a different role in the investors’ portfolios. They may also require different management techniques. By convention, financial securities can be distinguished into:

- money market securities
- capital market securities
- derivative securities.

You will find a description of the main types of financial securities in Chapter 2 of your textbook by Elton et al. Money market securities are short-term financial instruments. When they are issued, their maturity is at most one year. The main money market securities are treasury bills, repurchase agreements (or Repos), certificates of deposit, bankers’ acceptances, and commercial paper.

Capital market securities are those financial instruments whose maturity is greater than one year when they are issued. They include treasury bonds, corporate bonds, common stock (or equity), and preferred stock.

Derivative securities are so called because their value derives from the value of an underlying asset or security (e.g. a stock). Derivatives can also be written so that their value depends on a commodity, such as cocoa or oil. The most important derivative securities are futures and options.

In this course, you will study in detail the risk characteristics of the various types of financial securities, and you will examine the most appropriate risk management strategies for each category of security.

Reading

Please now read Chapter 2 of the textbook by Elton et al. This chapter contains a description of both money market and capital market securities, which form the main instruments of financial portfolios.

It is important that you familiarise yourself with the definitions of all these instruments, so make sure your notes cover these.
Chapter 3 of Elton et al. deals with the mechanics of financial markets. You will not need a detailed knowledge of the working of financial markets for this course on Risk Management. Hence, you do not need to study this chapter in great detail. There are, however, two topics which can be quite important in portfolio analysis and risk management. The first one is short sales. These consist of the sale of securities which investors do not own. These operations are usually intermediated by brokerage firms, so that the investor does not generally know the identity of the actual owner of the securities that are borrowed. Short sales are explained on pages 25–26 of Elton et al.

The second important topic is represented by the margins on levered investments. These consist of the deposit, or the cash amount, which is paid when purchasing securities (the remaining amount can be borrowed). They are subject to two types of regulations. These are the regulations that control

- the amount which investors can borrow when purchasing securities (initial margin requirements) and
- the extent to which the value of the margin can fall relative to the value of the assets before action must be taken to restore the margin at the appropriate level (maintenance margin requirements).

Margins are also important for futures and options investment. They are briefly explained by Elton and his colleagues on pages 27–30.

Reading

Please now skim quickly through Chapter 3 of the textbook by Elton et al. You should, however, pay close attention to the operations of short sales and margins, both of which are described in Chapter 3, on pages 25–26 and 27–30, respectively.

1.4 The Mean–Variance Approach

The mean–variance approach to portfolio analysis is originally due to Harry Markowitz, who developed it in the 1950s. Any investment in financial securities is associated with a fundamental uncertainty about its outcome. It is generally impossible to predict with certainty the actual return from an investment. What you can do is to characterise the uncertainty about the likely outcomes in terms of a probability distribution, which summarises your degree of belief about the likelihood of the possible returns. This probability distribution could be based on the past historical performance of the securities, possibly modified to reflect the investors’ knowledge of the current market conditions.

On the basis of the probability distribution of returns, you can compute the mean return, or expected return, on the securities, which is a measure of the ‘centre’ of the distribution. You could also compute the variance: this is a measure of the spread, or dispersion, of the securities about their mean value.
The square root of the variance, which is called the *standard deviation*, is also a widely used measure of dispersion.

The main practical difference between the standard deviation and the variance is that the former is expressed in the same unit of measurement as the returns, whereas the variance would be measured in ‘returns squared’. In computing the variance we take the deviations of each value from the mean, square them, and then take an average.

According to the mean–variance approach to portfolio analysis, all an investor needs to know about a portfolio of securities are the mean and the variance (or, equivalently, the standard deviation). Investors will have preferences over the mean and the variance of portfolios: they prefer a greater expected return to less, and (since they are assumed to be risk averse) less variance to more. Their utility function is thus an increasing function of the expected return, and a decreasing function of the variance of their investment. In principle, they may be happy to accept a greater risk (*i.e.* a larger variance for their investment) if this is associated with a sufficiently large increase in the expected return. The portfolio selection problem therefore goes as follows.

Investors compute the mean and the variance of all possible investment portfolios, and then select that portfolio which maximises their utility in terms of the mean–variance combination it offers.

All the relevant information for investors is thus summarised in the mean return from the investment and its variance. A more detailed knowledge of the distribution of returns would be irrelevant. This is clearly a very powerful simplification of the original problem of choice under uncertainty, since it enables investors to concentrate on just these two summary measures of the distribution of returns. Although this approach is not completely general (there could be instances, for example, in which investors are concerned with the possible *asymmetry* of the distribution, and this is not captured by the mean and the variance alone), in practice this simplification is usually regarded as satisfactory for a large class of investment problems, and has proven itself to be very useful in many actual applications.

**Reading**

Please now read Chapter 4 from the book by Elton *et al.*, pages 42–62.

As you read, make sure your notes enable you to answer the following questions:

- How can you determine the expected (or also mean or average) return of an asset given the probability distribution of its returns?
- How can you compute the variance and the standard deviation of the returns of an asset given the probability distribution of its returns?
How can you compute the expected return and the variance of a combination of assets?

What is the covariance between two assets? How can you interpret it? What does it mean if the covariance between two assets is positive or negative?

What happens when you form large portfolios? What are the main implications of diversification? What is the role of covariances?

1.4.1 **Mean and variance of one asset**

Suppose we consider $N$ financial assets, where the return on the $i$-th asset can take the $M$ values $R_{i1}, R_{i2}, \ldots, R_{iM}$ with probabilities $P_{i1}, P_{i2}, \ldots, P_{iM}$. The mean return, or expected return, on the $i$-th asset is given by

$$R_i = P_{i1}R_{i1} + P_{i2}R_{i2} + \ldots + P_{iM}R_{iM}$$

This value should be computed for each asset $i = 1, 2, \ldots, N$. The formula for the expected return can be expressed in a more compact form by using the summation symbol, $\sum$:

$$\bar{R}_i = \sum_{j=1}^{M} P_{ij} R_{ij} \quad i = 1, 2, \ldots, N$$

(1)

The variance of the $i$-th asset is given by

$$\sigma_i^2 = \sum_{j=1}^{M} P_{ij} (R_{ij} - \bar{R}_i)^2$$

(2)

The standard deviation, defined as the square root of the variance, is denoted by $\sigma_i$. The mean return is a measure of the *central tendency* of the distribution, and the variance and standard deviation are measures of the *spread around its centre*.

The summation symbol $\sum$ is a very useful tool to represent a summation in a compact fashion. Suppose you want to add together $n$ terms, $a_1, a_2, \ldots, a_i, \ldots, a_n$:

$$a_1 + a_2 + \ldots + a_i + \ldots + a_n$$

By using the summation symbol, you can simply write this sum as:

$$\sum_{i=1}^{n} a_i$$

1.4.2 **Mean, variance and covariance of a portfolio**

If you hold a combination, or portfolio, of assets you could similarly compute the expected return and the variance of the whole portfolio. Suppose assets $1, 2, \ldots, N$ are held in the proportions $X_1, X_2, \ldots, X_N$ respectively: then the expected return on the portfolio, denoted by $\bar{R}_p$, is given by

$$\bar{R}_p = E(R_p) = \sum_{i=1}^{N} X_i \bar{R}_i$$

(3)
where \(X_1 + X_2 + \ldots + X_N = 1\). When computing the variance, we also have to concern ourselves with how the asset returns *vary together, or covary*. If the returns tend to move in opposite directions (that is, when the returns on some of the assets are high, the returns on some others are usually low, and *vice versa*), then this produces the effect of reducing the overall variability of the portfolio. By contrast, if returns tend all to move in the same direction, then the variability of the portfolio is increased. The statistical measure of how two assets move together is given by their *covariance*: the covariance between assets 1 and 2 is denoted by \(\sigma_{12}\), and is defined as

\[
\sigma_{12} = \sum_{j=1}^{M} P_{12,j} (R_{1j} - \overline{R}_1)(R_{2j} - \overline{R}_2)
\]

where \(P_{12,j}\) denotes the joint probability that the return on the first asset is equal to \(R_{1j}\) and the return on the second asset is equal to \(R_{2j}\). You could similarly define the covariance between two generic assets \(i\) and \(k\), to be denoted by \(\sigma_{ik}\).

If \(\sigma_{12}\) is positive, when the return on the first asset is greater than the mean value \(\overline{R}_1\) (that is, \(R_{1j} - \overline{R}_1 > 0\)), then the return on the second asset is also, on average, greater than its mean value \(\overline{R}_2\) (that is, \(R_{2j} - \overline{R}_2 > 0\)).

Conversely, when \(R_{1j} - \overline{R}_1 < 0\), then we also have that, on average, \(R_{2j} - \overline{R}_2 < 0\). Thus, the returns on the two assets tend to move, on average, in the same direction.

By contrast, if \(\sigma_{12}\) is negative, the returns on assets 1 and 2 tend to move in opposite directions and to offset each other: when the return on the first asset is greater than its mean value \(\overline{R}_1\) (that is, \(R_{1j} - \overline{R}_1 > 0\)), then the return on the second asset is, on average, less than its mean value \(\overline{R}_2\) (that is, \(R_{2j} - \overline{R}_2 < 0\), and *vice versa*. The fact that the returns on the assets move in opposite directions reduces the variability of the portfolio. In fact, the portfolio tends to be insulated from shocks to the returns on the assets which form it.

In general, the variance of a portfolio of assets is given by

\[
\sigma_p^2 = \text{Var}(R_p) = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{k=1}^{N} X_i X_k \sigma_{ik}
\]

and its standard deviation is \(\sigma_p = SD(R_p) = \sqrt{\sigma_p^2}\). A measure of the association between two assets, which is always in the range between \(-1\) and \(+1\), is the *correlation coefficient*. The correlation coefficient between assets 1 and 2, denoted as \(\rho_{12}\), is defined as

\[
\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}
\]

where \(\sigma_1 = \sqrt{\sigma_1^2}\) and \(\sigma_2 = \sqrt{\sigma_2^2}\) are the standard deviations of assets 1 and 2 respectively. The correlation coefficient always has the same sign as the
covariance. Thus, if \( \rho_{12} > 0 \) then the returns on assets 1 and 2 will tend to move, on average, in the same direction. If \( \rho_{12} < 0 \), they will tend to move in opposite directions. Furthermore, it is always true that \(-1 \leq \rho_{12} \leq 1\).

\[ \rho_{12} > 0 \] then the returns on assets 1 and 2 will tend to move, on average, in the same direction. If \( \rho_{12} < 0 \), they will tend to move in opposite directions. Furthermore, it is always true that \(-1 \leq \rho_{12} \leq 1\).

\[ \rho_{12} < 0 \] then the returns on assets 1 and 2 will tend to move, on average, in the same direction. If \( \rho_{12} > 0 \), they will tend to move in opposite directions. Furthermore, it is always true that \(-1 \leq \rho_{12} \leq 1\).

Exercise

Now please solve problems 1 and 2 on pages 62–63 of the textbook by Elton et al. These problems require you to compute mean returns, standard deviations, covariances and correlation coefficients for assets and for portfolios of assets. It is important that you familiarise yourselves with these computations. This is also a good way to make sure that you have really understood the theory. Answers are provided at the end of this unit.

1.4.3 Variance of a portfolio: diversification and risk

What happens to the variance of a portfolio as the number of assets gets large? Look again at the formula for the variance, \( \sigma_p^2 \), and assume that all the \( N \) assets are held in equal proportions: this means that \( X_i = 1/N \), for all \( i \). The variance becomes

\[ \sigma_p^2 = \sum_{i=1}^{N} \frac{1}{N^2} \sigma_i^2 + \sum_{i=1}^{N} \sum_{k=1}^{N} \frac{1}{N^2} \sigma_{ik} \quad (7) \]

We could try to write the above formula in terms of the average variance for the assets and of their average covariance. These can be defined as

\[ \bar{\sigma}_j^2 = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2 \quad \bar{\sigma}_{ik} = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{k=1}^{N} \sigma_{ik} \quad (8) \]

(Note that there are \( N(N-1) \) elements inside the double summation symbol in the formula for \( \bar{\sigma}_{ik} \), since each of the \( N \) assets can be combined with any of the remaining \( N-1 \) assets to compute the covariance.) The variance of the portfolio can therefore be written as

\[ \sigma_p^2 = \frac{1}{N} \bar{\sigma}_i^2 + \frac{N-1}{N} \bar{\sigma}_{ik} \quad (9) \]

We can now see what happens to \( \sigma_p^2 \) as the number of assets gets large (i.e. as \( N \to \infty \)). The coefficient \( 1/N \) on \( \bar{\sigma}_i^2 \) tends to become negligible, whereas the coefficient \( (N-1)/N \) on \( \bar{\sigma}_{ik} \) tends to unity (the numerator is about equal to the denominator, when \( N \) is a large number). Thus, as the number \( N \) of assets in a portfolio increases, the importance of the variances of the individual assets tends to vanish (since \( 1/N \to 0 \)), and the variance of the portfolio converges to the average covariance between the assets (since \( (N-1)/N \to 1 \)).

Formally, we write

\[ \sigma_p^2 \to \bar{\sigma}_{ik} \quad \text{as} \quad N \to \infty \]
In the jargon of portfolio analysis, individual risk is completely diversified away: individual variances play no role whatsoever in a large portfolio. Elton et al. provide an example of this result in their Table 4.8, page 57.

### 1.5 The Opportunity Set under Risk – Efficient Portfolios

The previous section has illustrated how you can summarise the uncertain returns on an asset in terms of its mean and variance (or, equivalently, the mean and the standard deviation). You have also seen how to compute the mean and the variance of the returns on a portfolio that comprises a collection of assets. The fundamental important result that was obtained in the previous section is that the properties of a portfolio can be very different from those of the underlying assets, when considered individually. In particular, in large portfolios the variability of individual assets is completely diversified away, and the risk of the portfolio only depends on the average covariance between the pairs of assets.

How can you use these results to construct ‘good’ portfolios? Ideally, you would like to form portfolios with a large expected return and a small variance.

The first step towards forming portfolios of financial assets is to analyse the combinations of a limited number of risky assets, in order to examine how the properties of the combinations of assets are related to the underlying securities. You will then be in a position to evaluate more complex portfolios, involving a large number of assets.

We have already established the importance of the covariances and correlation coefficients when looking at the variability of the returns on a portfolio. In the present section I will ask you to consider simple portfolios obtained by combining two assets only, and to examine how the properties of the resulting combinations critically depend on the correlation coefficient between the underlying assets.

**Reading**

The material covered in this section is presented in the first sections of Chapter 5 of Elton et al., pages 65–74. Please read those pages now. You should make sure you can follow all the algebraic steps of the presentation.

Suppose you consider two assets, A and B, with expected returns $\bar{R}_A = E(R_A)$ and $\bar{R}_B = E(R_B)$ respectively, with variances $\sigma_A^2$ and $\sigma_B^2$, and with covariance $\sigma_{AB}$. If you form a portfolio $P$ in which a proportion $X_A$ is invested in asset A and a proportion $X_B = (1 - X_A)$ is invested in asset B, the expected return of the portfolio $P$ is
\[ \bar{R}_p = X_A \bar{R}_A + (1 - X_A) \bar{R}_B \]  
(10)

and the variance is

\[ \sigma_p^2 = X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2 + 2X_A(1 - X_A)\rho_{AB}\sigma_A\sigma_B \]  
(11)

The standard deviation can be written, using the equality \( \rho_{AB} = \sigma_{AB}/\sigma_A\sigma_B \), as

\[ \sigma_p = [X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2 + 2X_A(1 - X_A)\rho_{AB}\sigma_A\sigma_B]^{1/2} \]  
(12)

These are extremely useful results. The expected return from the portfolio depends on the expected returns on the underlying assets, \( \bar{R}_A \) and \( \bar{R}_B \), and on the relative proportions in which they are held, \( X_A \) and \( (1 - X_A) \). By contrast, the risk associated with the portfolio \( P \) is not only related to the variability of the underlying risky assets, as measured by their variances \( \sigma_A^2 \) and \( \sigma_B^2 \), and to their relative proportions, \( X_A \) and \( (1 - X_A) \), but also to their correlation coefficient, \( \rho_{AB} \). In particular, if the correlation coefficient between \( A \) and \( B \) is negative, the risk of the portfolio will be reduced. This should be an intuitive result: if \( \rho_{AB} < 0 \), then the returns on \( A \) and \( B \) will tend to move in opposite directions, and will partially offset each other. In other words, when the return on one of the assets is high, the return on the other asset will on average be low. As a result, the portfolio formed of \( A \) and \( B \) is less variable than the underlying individual assets, because the movements in the returns tend to cancel each other out.

The ability of investors to reduce the variability of their position by diversifying therefore depends on the correlation coefficients between the assets being relatively low, or even negative. If the correlation coefficient is close to +1, then the returns on the assets will always tend to vary in the same direction, and the risk-reduction advantages from diversification are negligible.

Elton and his colleagues distinguish four cases, depending on whether the correlation coefficient is equal to +1, –1, 0 or 0.5. In the first case, the standard deviation of the portfolio \( P \) is:

\[ \sigma_p = X_A \sigma_A + (1 - X_A) \sigma_B \]  
(13)

(Elton et al., page 67). When \( \rho_{AB} = +1 \), there is no risk reduction from diversification: the standard deviation of the portfolio is simply the weighted average of the standard deviations of \( A \) and \( B \). Figure 1.1 below illustrates this case. At point \( A \), the portfolio only includes asset \( A \): that is, \( X_A = 1 \). At point \( B \), the portfolio only includes asset \( B \): \( X_B = 1 \), and therefore \( X_A = 0 \). As \( X_A \) decreases from 1 to 0, the point on the graph representing the portfolio moves along the straight line from point \( A \) to point \( B \).
When $\rho = -1$, by contrast, the variance of the portfolio is
\[
\sigma_P^2 = [X_A \sigma_A - (1 - X_A) \sigma_B]^2
\]  
(14)
and the standard deviation of the portfolio is:
\[
\sigma_P = \begin{cases} 
X_A \sigma_A - (1 - X_A) \sigma_B & \text{if } X_A \sigma_A - (1 - X_A) \sigma_B \geq 0 \\
(1 - X_A) \sigma_B - X_A \sigma_A & \text{if } X_A \sigma_A - (1 - X_A) \sigma_B \leq 0
\end{cases}
\]  
(15)
(Elton et al., page 69). The reason for having two separate analytical expressions for the standard deviation is that we have to make sure that $\sigma_P$ is defined to be non-negative, when we take the square root of the variance.

The important result is that we can now reduce the variability of our investment by diversifying. In particular, we can form a portfolio with zero risk. This is obtained by setting the standard deviation equal to zero: $\sigma_P = 0$. This yields
\[
X_A = \frac{\sigma_B}{\sigma_A + \sigma_B}
\]  
(16)
and thus also
\[
X_B = 1 - X_A = \frac{\sigma_A}{\sigma_A + \sigma_B}
\]  
(17)
When the assets $A$ and $B$ are held in exactly the above proportions, the portfolio $P$ will have zero variance: hence, the return on $P$ will always be equal to the expected return, $\overline{R}_p$. In this case, the advantages from diversification are potentially very large, since you will be able to completely eliminate the risk from your portfolio of (risky) assets.

This situation is illustrated by Figure 1.2.
At point $A$, the portfolio only includes asset $A$: that is, $X_A = 1$. At point $B$, the portfolio only includes asset $B$: $X_B = 1$, and therefore $X_A = 0$. As $X_A$ decreases from 1 to 0, the point on the graph representing the portfolio moves along the broken straight line from $A$ to $B$. When $X_A = \sigma_B/(\sigma_A + \sigma_B)$, the portfolio is described by point $Z$: this is the zero-variance portfolio. It lies on the vertical axis, since its variance (and hence also its standard deviation) is equal to zero.

It is important to note that all portfolios on the lower segment of the line from $A$ to $Z$ have the same standard deviation, but lower expected return, compared to portfolios that lie on the upper segment of the line, from $Z$ to $B$. For instance, the portfolio represented by point $E$ has the same risk (in terms of the standard deviation), but a lower expected return, compared to the portfolio described by point $F$. If you are a rational, risk-averse investor, you will never choose to hold portfolios on the line from $A$ to $Z$: these portfolios are inefficient. The reason is that these portfolios are dominated by portfolios on the upper segment, which offer a higher expected return for a given standard deviation. The only efficient portfolios are thus on the line from $Z$ to $B$: they form the set of efficient portfolios.

In practice, the correlation coefficient will usually be a number between $-1$ and +1. When $\rho = 0$, for instance, the standard deviation of the portfolio is

$$\sigma_p = \left[ X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2 \right]^{1/2}$$

(see also Elton et al., page 71). The set of portfolios formed by combining assets $A$ and $B$ is now described not by a straight line, but by a curve on the $(\sigma_p, \bar{R_p})$ plane (Figure 1.3).
Diversification can still reduce the risk of the portfolio, but we can no longer construct a zero-variance portfolio (we can only do this when $\rho = -1$). However, we can still find a minimum-variance portfolio. This is obtained by selecting that value of $X_A$ for which the variance (or, equivalently, the standard deviation) is smallest. This problem can be solved by taking the derivative of $\sigma_P^2$ (or of $\sigma_P$) with respect to $X_A$ and setting it equal to 0. This is shown in detail by Elton et al. (pages 71–74) for the general case in which $-1 < \rho < 1$; when $\rho = 0$, the critical value of $X_A$ for which the variance $\sigma_P^2$ is minimised is

$$X_A = \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}$$

This is shown as point $C$ in Figure 1.4 below.

The portfolios that lie on the line from $A$ to $C$ are inefficient, since for each one of them we can find an alternative portfolio, on the line from $C$ to $B$, which has the same standard deviation but a higher expected return (by the same reasoning we used for Figure 1.2). The set of efficient portfolios is now formed of the line from point $C$ to point $B$.

Thus, given two risky assets $A$ and $B$, we can represent the set of all their possible portfolios as a curve connecting the two assets on the standard deviation–expected return plane. This has been illustrated for $\rho = +1$, $\rho = -1$ and $\rho = 0$. When $\rho = +1$, the investment possibilities are a straight line connecting the assets $A$ and $B$. When $\rho = -1$, the investment possibilities are a broken straight line, which intersects the vertical axis at the zero-variance portfolio. These are the two extreme cases: for intermediate values of the correlation coefficient (that is, when $\rho$ is strictly greater than $-1$ and strictly
less than +1) the portfolio possibilities will be described by a concave curve connecting the assets (as in Figures 1.3 and 1.4).

**Figure 1.4**

![Figure 1.4](image)

**Exercise**

Please now solve problem 1 part A on page 92 of the textbook by Elton et al. An answer is provided at the end of this unit.

### 1.6 Short Sales and Riskless Lending and Borrowing

There are two important extensions to the analysis carried out in the previous section. First, you have to consider what happens when you are allowed to sell short some of your assets – that is, you can sell assets that you do not yet own. Second, you have to describe your portfolio possibilities when you can lend at a riskless rate (for instance, if you can purchase very short-term government bills), or borrow at the riskless rate.

Let us start by considering the possibility of short-selling assets (or selling assets short). If you have two risky assets, $A$ and $B$, the portfolio possibility set can be described by a curve connecting them, as in Figure 1.5. At point A, 100% of wealth is invested in asset $A$, and at point B, 100% of wealth is invested in asset $B$. However, if short sales are allowed, investors can sell short asset $A$, and increase their holdings of asset $B$. If they do that, they will be able to move to the right of point B on the curve in the figure. In fact, more than 100% of their original wealth is invested in asset $B$. They are able to invest an amount greater than their wealth, because they have sold short an asset ($A$) they do not own. Similarly, they could sell short asset $B$, increase
their wealth, and invest everything in asset A. If they do that, they will hold a portfolio on the curve to the right of point A.

**Figure 1.5**

![Portfolio Possibility Curve](image)

The total portfolio possibility set when short sales are allowed is thus the line going through A to B, but extending to the right of each point. From Figure 1.5, you can also see that the portfolios in the lower section of the curve from point C (including point A) are inefficient, since they are dominated by portfolios on the upper half of the curve. The latter portfolios have a higher expected return for any given standard deviation, and would therefore be chosen by rational investors who prefer more to less and who are risk averse. Hence, the set of efficient portfolios when short sales are allowed is the upper half of the curve from point C. Rational and risk-averse investors will select a portfolio from the efficient set.

**Reading**

Please now read the section of Chapter 5, Elton et al. on pages 74–81. The authors first discuss the feasible shapes of the portfolio possibility curve in the absence of short sales, and then introduce short sales and examine how this process enhances the possibility set.

As you read, please make sure you can answer the following questions.

- Why is the portfolio possibilities curve convex (with respect to the vertical axis)?
- Why do short sales increase the investment possibilities?
- What are the cash flows for an investor who has sold short financial securities?

Our next extension involves the possibility of investing in a safe asset. This can be regarded as a ‘sure’ investment – that is, an investment which delivers a promised return with certainty. Formally, the variance of its returns is equal to zero. The return on the asset is thus always equal to a constant value. As a
consequence, this asset can be represented as a point on the vertical axis, on the standard deviation–expected return plane (point $F$ in Figure 1.6).

**Figure 1.6**

What happens when we form a portfolio which includes the safe asset, $F$, together with a risky asset, $A$? In this case, the portfolio can be described as a point on the straight line connecting $F$ to $A$ (Figure 1.6). The expected return on the portfolio is given by

$$\bar{R}_p = (1 - X_A) R_F + X_A \bar{R}_A$$

(20)

where $R_F$ is the return on the safe asset, $\bar{R}_A$ is the expected return on the risky asset, and $X_A$ is the proportion of the risky asset held in the portfolio. Since the variance of the safe asset $F$ is equal to zero, the variance of the return on the portfolio is simply:

$$\sigma_p^2 = X_A^2 \sigma_A^2$$

(21)

and therefore the standard deviation is:

$$\sigma_p = X_A \sigma_A$$

(22)

Investing in the safe asset $F$ which gives a return $R_F$ with certainty is equivalent to lending at the safe rate of interest $R_F$. If short sales are allowed, we could sell short the safe asset $F$: this would move us to the right of point $A$ in Figure 1.6, and would be equivalent to borrowing at the safe rate of interest $R_F$.

### Reading

Please read now the rest of Chapter 5, pages 81–92, which first describes the analytics of lending and borrowing in the presence of a safe rate of return, and then illustrates a few examples of the efficient frontier. It is important that you read the text and the examples carefully.
1.7 How to Compute the Efficient Set

The previous two sections discussed the shape of the portfolio possibilities of risky assets, with or without the presence of a risk-free (or safe) asset. When you can choose among many risky assets, and no pair of them is perfectly negatively correlated, then the set of all the portfolio possibilities is represented by a convex set, and the set of efficient portfolios by a concave line (the upper north-west frontier of the possibility set) on the \((\sigma, \overline{R})\) plane.

The exact choice of the investors will then be determined by their preferences, according to the mean–variance approach. An illustration of this is given in Figure 1.7.

**Figure 1.7**

The shaded area of the convex set \(ACB\) describes the portfolio possibilities, the line \(CB\) is the set of efficient portfolios, and \(U_0, U_1\) are indifference curves for the investors.

An indifference curve measures risk-return combinations that yield the same level of overall utility to the investor. They are positively sloped, since investors require a higher expected return to compensate for a higher standard deviation. The curve \(U_1\) corresponds to a higher utility level than \(U_0\), since it corresponds to a higher expected return for any given standard deviation. The optimal choice for the investor is point \(D\), where the indifference curve \(U_1\) is tangent to the efficient frontier of the portfolio possibilities set.
But how can we find the efficient frontier \( CB \) of the set of portfolios, given the assets that are available?

The exact mathematical solution to this problem can be quite complicated to work out, and we shall not go into the details of the computations. In practice, investors use computer algorithms to find the solution to these problems. Elton and his co-authors outline a general approach to the problem in Chapter 6 of their textbook, and explain how to express the selection of the optimal portfolio in exact mathematical terms. This could be useful for two reasons. Firstly, you can understand how computer programs are constructed. Secondly, and more importantly for this course, expressing the problem in mathematical terms makes it apparent just how much information on the assets is required in order to find the efficient set. We need to know not just the expected values and variances of each asset, but also all the pairwise correlation coefficients. These informational requirements can be quite formidable, when the number of assets considered is even moderately large. Thus, the next units will look at some possible ways to simplify this problem.

### Optional Reading

If you wish, you can now read Chapter 6 on ‘Techniques for Calculating the Efficient Frontier’ from Elton et al., pages 95–106. This chapter is optional reading, and you can omit it without prejudice for the rest of the course.

### 1.8 Conclusions

This unit has described the fundamentals of the mean–variance approach. You have seen:

- how to compute the expected value and the variance of a financial portfolio, and
- how to compute the covariance between a pair of securities.

You have also examined

- the benefits from diversification,

and have seen how

- the variance of individual assets tends to have a negligible importance in large portfolios.

You have also studied

- the shape of the opportunity set under risk.

The next units will present some ways to further simplify the mean–variance approach, and will put forward some operationally feasible methods for constructing financial portfolios.
References


Answers to Questions

Chapter 4, Exercise 1 (Elton et al., pages 62–63)

The calculations and plot are in Excel file (1997-2003 compatible) C323_U1_Elton_Chapter 4_Q1.xls

A

\[ E(R_1) = 12 \quad s_1 = 2.83 \]
\[ E(R_2) = 6 \quad s_2 = 1.41 \]
\[ E(R_3) = 14 \quad s_3 = 4.24 \]
\[ E(R_4) = 12 \quad s_4 = 3.27 \]

B

\[ s_{12} = -4 \quad s_{13} = 12 \quad s_{14} = 0 \]
\[ s_{23} = -6 \quad s_{24} = 0 \quad s_{34} = 0 \]

C

\[ E(R_a) = 9 \quad \sigma_a^2 = 0.5 \]
\[ E(R_b) = 13 \quad \sigma_b^2 = 12.5 \]
\[ E(R_c) = 12 \quad \sigma_c^2 = 4.6666 \ldots = 4.6 \]
\[ E(R_d) = 10 \quad \sigma_d^2 = 2 \]
\[ E(R_e) = 13 \quad \sigma_e^2 = 7.16 \]
\[ E(R_f) = 10.6 \quad \sigma_f^2 = 3.53 \]
\[ E(R_g) = 10.6 \quad \sigma_g^2 = 2.07 \]
\[ E(R_h) = 12.6 \quad \sigma_h^2 = 6.74 \]
\[ E(R_i) = 11 \quad \sigma_i^2 = 2.66 \]

Chapter 4, Exercise 2 (Elton et al., page 63)

See the Excel file C323_U1_Elton_Chapter 4_Q2.xls for details of the calculations.

A

<table>
<thead>
<tr>
<th>Time</th>
<th>RA</th>
<th>RB</th>
<th>RC</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.0368</td>
<td>0.1051</td>
<td>0.0141</td>
</tr>
<tr>
<td>3</td>
<td>0.0038</td>
<td>0.0050</td>
<td>0.1492</td>
</tr>
<tr>
<td>4</td>
<td>-0.0653</td>
<td>0.0373</td>
<td>-0.0141</td>
</tr>
<tr>
<td>5</td>
<td>0.0135</td>
<td>0.0098</td>
<td>0.1084</td>
</tr>
<tr>
<td>6</td>
<td>0.0618</td>
<td>0.0339</td>
<td>0.0492</td>
</tr>
<tr>
<td>7</td>
<td>0.0212</td>
<td>-0.0145</td>
<td>0.1693</td>
</tr>
</tbody>
</table>

B

\[ E(R_A) = 0.0120 \quad E(R_B) = 0.0295 \quad E(R_C) = 0.0793 \]

C

\[ \sigma_A = SD(R_A) = 0.0392 \quad \sigma_B = 0.0381 \quad \sigma_C = 0.0680 \]
\( \rho_{AB} = \text{Corr}(R_A, R_B) = 0.1406 \quad \rho_{AC} = 0.2751 \quad \rho_{BC} = -0.77435 \)

**E Portfolio**

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>( E(R_P) )</th>
<th>( \sigma_P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2} A + \frac{1}{2} B )</td>
<td>0.0207</td>
<td>0.0292</td>
</tr>
<tr>
<td>( \frac{1}{2} A + \frac{1}{2} C )</td>
<td>0.0457</td>
<td>0.0437</td>
</tr>
<tr>
<td>( \frac{1}{2} B + \frac{1}{2} C )</td>
<td>0.0544</td>
<td>0.0227</td>
</tr>
<tr>
<td>( \frac{1}{3} A + \frac{1}{3} B + \frac{1}{3} C )</td>
<td>0.0403</td>
<td>0.0247</td>
</tr>
</tbody>
</table>

**Chapter 5, Exercise 1 (Elton et al., page 92)**

The calculations and graphs are included in Excel file C323_U1_Elton_Chapter 5_Q1.xls

**A** The correlation between returns for securities 1 and 2 is \(-1\). In which case the minimum variance portfolio has zero variance, and the weights in such a portfolio are obtained from equations (16) and (17). The graph of expected portfolio return against portfolio standard deviation is obtained by varying the weights on security 1 from 0 to 1, in increments of 0.05, and calculating the expected returns and standard deviation. The zero variance portfolio is also included in the graph in the Excel file. You should obtain a graph similar to Figure 1.2.

\( (1) \quad X_1 = 1/3 \quad X_2 = 2/3 \quad E(R_P) = 8 \quad \sigma_P = 0 \)

(4) **Assets 1 and 3:**

The correlation coefficient for the returns on assets 1 and 3 is \(+1\). Therefore the minimum variance portfolio is obtained by investing in the asset with the lower variance. The graph of portfolio expected return against standard deviation for various weights for asset 1 and asset 3 is again obtained by varying \( X_1 \) from 0 to 1 in increments of 0.05. It is similar to Figure 1.1.

\( X_1 = 1 \quad X_3 = 0 \quad E(R_P) = 12 \quad \sigma_P = 2.8284 \)

**Assets 1 and 4:**

The returns on assets 1 and 4 are independent, and therefore the covariance (and correlation coefficient) equals zero. The minimum variance portfolio is obtained by minimising the variance with respect to the weight of one of the assets; when the correlation coefficient is zero this leads to equation (19). In most cases you would expect to obtain a graph of expected portfolio return against portfolio standard deviation similar to Figure 1.4. However, the expected return for asset 1 equals the expected return for asset 4 (equals 12), so the graph of expected return against standard deviation for the portfolio (where the weights always sum to 1) is a horizontal straight line.
$X_1 = 0.5714 \quad X_4 = 0.4286 \quad E(R_P) = 12 \quad \sigma_P = 2.1381$

**Assets 2 and 3:**
The correlation coefficient between the returns on assets 2 and 3 is $-1$. It is possible to achieve a portfolio with zero variance, and the weights are obtained from equations (16) and (17). You should obtain a graph of portfolio expected return against standard deviation similar to Figure 1.2.

$X_2 = 0.75 \quad X_3 = 0.25 \quad E(R_P) = 8 \quad \sigma_P = 0$

**Assets 2 and 4:**
The correlation coefficient between returns on asset 2 and asset 4 is zero. The weights for the minimum variance portfolio are obtained from equation (19), and you should obtain a plot of portfolio expected return against standard deviation similar to Figure 1.4.

$X_2 = 0.8421 \quad X_4 = 0.1579 \quad E(R_P) = 6.9474 \quad \sigma_P = 1.2978$

**Assets 3 and 4:**
The correlation coefficient between the returns on asset 3 and asset 4 is also zero, and the same arguments and methods can be applied.

$X_3 = 0.3721 \quad X_4 = 0.6279 \quad E(R_P) = 12.7442 \quad \sigma_P = 2.5880$

**Chapter 5, Exercise 5 (Elton et al., page 93)**
The Excel file C323_U1_Elton_Chapter 5_Q5.xls contains relevant calculations and plots. The plots of portfolio expected return against standard deviation are obtained by varying one weight from 0 to 1 in increments of 0.05.

$\rho = 1$:
When $\rho = 1$ the minimum variance portfolio is obtained by investing only in the asset with the lower variance. You should obtain a plot of portfolio expected return against standard deviation similar to Figure 1.1.

$X_1 = 0 \quad X_2 = 1 \quad \sigma_P = 2$

$\rho = 0$:
For $\rho = 0$ the weights in the minimum variance portfolio are obtained from equation (19), and you should obtain a plot of portfolio expected return against standard deviation similar to Figure 1.4.

$X_1 = 0.1379 \quad X_2 = 0.8621 \quad \sigma_P = 1.8570$

$\rho = -1$:
For $\rho = -1$ the minimum variance portfolio has zero variance; the weights are obtained from equations (16) and (17); and the graph of portfolio expected return against standard deviation should be similar to Figure 1.2.

$X_1 = 0.2857 \quad X_2 = 0.7143 \quad \sigma_P = 0$