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The Arab World edition of *Introductory Mathematical Analysis for Business, Economics, and the Life and Social Sciences* is built upon one of the finest books of its kind. This edition has been adapted specifically to meet the needs of students in the Arab world, and provides a mathematical foundation for students in a variety of fields and majors. It begins with precalculus and finite mathematics topics such as functions, equations, mathematics of finance, matrix algebra, linear programming, and probability. Then it progresses through both single variable and multivariable calculus, including continuous random variables. Technical proofs, conditions, and the like are sufficiently described but are not overdone. Our guiding philosophy led us to include those proofs and general calculations that shed light on how the corresponding calculations are done in applied problems. Informal intuitive arguments are often given as well.

**Approach**

The Arab World Edition of *Introductory Mathematical Analysis for Business, Economics, and the Life and Social Sciences* follows a unique approach to problem solving. As has been the case in earlier editions of this book, we establish an emphasis on algebraic calculations that sets this text apart from other introductory, applied mathematics books. The process of calculating with variables builds skill in mathematical modeling and paves the way for students to use calculus. The reader will not find a “definition-theorem-proof” treatment, but there is a sustained effort to impart a genuine mathematical treatment of real world problems. Emphasis on developing algebraic skills is extended to the exercises, in which many, even those of the drill type, are given with general coefficients.

In addition to the overall approach to problem solving, we aim to work through examples and explanations with just the right blend of rigor and accessibility. The tone of the book is not too formal, yet certainly not lacking precision. One might say the book reads in a relaxed tone without sacrificing opportunities to bring students to a higher level of understanding through strongly motivated applications. In addition, the content of this edition is presented in a more logical way for those teaching and learning in the Arab region, in very manageable portions for optimal teaching and learning.

**What's New in the Arab World Edition?**

A number of adaptations and new features have been added to the Arab World Edition.

- **Additional Examples and Problems:** Hundreds of real life examples and problems about the Arab World have been incorporated.
- **Additional Applications:** Many new *Apply It* features from across the Arab region have been added to chapters to provide extra reinforcement of concepts, and to provide the link between theory and the real world.
- **Chapter test:** This new feature has been added to every chapter to solidify the learning process. These problems do not have solutions provided at the end of the book, so can be used as class tests or homework.
- **Biographies:** These have been included for prominent and important mathematicians. This historical account gives its rightful place to both Arab and international contributors of this great science.
- **English-Arabic Glossary:** Mathematical, financial and economic terms with translation to Arabic has been added to the end of the book. Any instructor with experience in the Arab World knows how helpful this is for the students who studied in high school in Arabic.
Other Features and Pedagogy

- **Applications:** An abundance and variety of new and additional applications for the Arab audience appear throughout the book; students continually see how the mathematics they are learning can be used in familiar situations, providing a real-world context. These applications cover such diverse areas as business, economics, biology, medicine, sociology, psychology, ecology, statistics, earth science, and archaeology. Many of these real-world situations are drawn from literature and are documented by references, sometimes from the Web. In some, the background and context are given in order to stimulate interest. However, the text is self-contained, in the sense that it assumes no prior exposure to the concepts on which the applications are based. (See, for example, page XXX, Example X in X.X)

- **Apply It:** The Apply It exercises provide students with further applications, with many of these covering companies and trends from across the region. Located in the margins, these additional exercises give students real-world applications and more opportunities to see the chapter material put into practice. An icon indicates Apply It problems that can be solved using a graphing calculator. Answers to Apply It problems appear at the end of the text and complete solutions to these problems are found in the Solutions Manuals. (See, for example, page XXX, Apply It X in X.X)

- **Now Work Problem N:** Throughout the text we have retained the popular Now Work Problem N feature. The idea is that after a worked example, students are directed to an end of section problem (labeled with a blue exercise number) that reinforces the ideas of the worked example. This gives students an opportunity to practice what they have just learned. Because the majority of these keyed exercises are odd-numbered, students can immediately check their answer in the back of the book to assess their level of understanding. The complete solutions to these exercises can be found in the Student Solutions Manual. (See, for example, page XXX, Example X in XX.X)

- **Cautions:** Throughout the book, cautionary warnings are presented in very much the same way an instructor would warn students in class of commonly-made errors. These Cautions are indicated with an icon to help students prevent common misconceptions. (See, for example, page XXX, Example X in XX.X)

- **Definitions, key concepts, and important rules and formulas** are clearly stated and displayed as a way to make the navigation of the book that much easier for the student. (See, for example, page XXX, Definition of Derivative in XX.X)

- **Explore & Extend Activities:** Strategically placed at the end of the chapter, these bring together multiple mathematical concepts studied in the previous sections within the context of a highly relevant and interesting application. Where appropriate, these have been adapted to the Arab World. These activities can be completed in or out of class either individually or within a group. (See, for example, page XXX, in Chapter XX)

- **Review Material:** Each chapter has a review section that contains a list of important terms and symbols, a chapter summary, and numerous review problems. In addition, key examples are referenced along with each group of important terms and symbols. (See, for example, page XXX, in Chapter XX)

- **Back-of-Book Answers:** Answers to odd-numbered problems appear at the end of the book. For many of the differentiation problems, the answers appear in both “unsimplified” and “simplified” forms. This allows students to readily check their work. (See, for example, page AN-XX, in Answers for XX.X)

Examples and Exercises

Most instructors and students will agree that the key to an effective textbook is in the quality and quantity of the examples and exercise sets. To that end, hundreds examples are worked out in detail. Many of these are new and about the Arab World, with real regional data and statistics included wherever possible. These problems take the reader from the population growth of Cairo, to the Infant Mortality rate in Tunisia, the life expectancy in Morocco, the
divorce rate in Algeria, the unemployment rate in Saudi Arabia, the exports and imports of Kuwait, the oil production in Tunisia and Saudi Arabia, Labor Force in Morocco, the CPI of Libya, the GDC of Lebanon, the population of Bahrain in the age group of 15 to 64, and the number of doctors in Jordan. They also include popular products from the region, and local companies like Air Arabia, Royal Jordanian Airline, Emirates, oil companies such as Aramco, postal companies like Aramex, telecommunication providers such as Etisalat or Menatel, the stocks of Emaar. Regional trends are also covered in these problems, such as internet users in Yemen, mobile subscriptions in Syria, the emission of CO2 in Qatar, the number of shops in Dubai, the production of oil and natural gas in Oman, the production of electricity and fresh orange in Morocco, the participation to the Olympic games by the Arab nations, and the concept of Murabaha in Islamic finance.

Some examples include a strategy box designed to guide students through the general steps of the solution before the specific solution is obtained (See pages XXX–XXX; XX.X example X). In addition, an abundant number of diagrams and exercises are included. In each exercise set, grouped problems are given in increasing order of difficulty. In most exercise sets the problems progress from the basic mechanical drill-type to more interesting thought-provoking problems. The exercises labeled with a blue exercise number correlate to a “Now Work Problem N” statement and example in the section.

A great deal of effort has been put into producing a proper balance between the drill-type exercises and the problems requiring the integration and application of the concepts learned. (see pages XXX–XXX; Explore and Extend for Chapter X; XXX. Explore and Extend for Chapter X; XXX–XXX, Example X in XX.X on Lines of Regression)

**Technology**

In order that students appreciate the value of current technology, optional graphing calculator material appears throughout the text both in the exposition and exercises. It appears for a variety of reasons: as a mathematical tool, to visualize a concept, as a computing aid, and to reinforce concepts. Although calculator displays for a TI-83 Plus accompany the corresponding technology discussion, our approach is general enough so that it can be applied to other graphing calculators. In the exercise sets, graphing calculator problems are indicated by an icon. To provide flexibility for an instructor in planning assignments, these problems are typically placed at the end of an exercise set.

**Course Planning**

One of the obvious assets of this book is that a considerable number of courses can be served by it. Because instructors plan a course outline to serve the individual needs of a particular class and curriculum, we will not attempt to provide detailed sample outlines. *Introductory Mathematical Analysis* is designed to meet the needs of students in Business, Economics, and Life and Social Sciences. The material presented is sufficient for a two semester course in *Finite Mathematics* and *Calculus*, or a three semester course that also includes *College Algebra* and *Core Precalculus* topics. The book consists of three important parts:

**Part I: College Algebra**

The purpose of this part is to provide students with the basic skills of algebra needed for any subsequent work in Mathematics. Most of the material covered in this part has been taught in high school.

**Part II: Finite Mathematics**

The second part of this book provides the student with the tools he needs to solve real-world problems related to Business, Economic or Life and Social Sciences.

**Part III: Applied Calculus**

In this last part the student will learn how to connect some Calculus topics to real life problems.
Supplements

- The *Student Solutions Manual* includes worked solutions for all odd-numbered problems and all *Apply It* problems. ISBN XXXXX | XXXXX

- The *Instructor’s Solution Manual* has worked solutions to all problems, including those in the *Apply It* exercises and in the *Explore & Extend* activities. It is downloadable from the Instructor’s Resource Center at XXXXX.

- *TestGen®*([www.pearsoned.com/testgen](http://www.pearsoned.com/testgen)) enables instructors to build, edit, and print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text. TestGen is algorithmically based, allowing instructors to create multiple but equivalent versions of the same question or test with the click of a button. Instructors can also modify test bank questions or add new questions. The software and testbank are available for download from Pearson Education’s online catalog and from the Instructor’s Resource Center at XXXXX.

- *MyMathLab*, greatly appreciated by instructors and students, is a powerful online learning and assessment tool with interactive exercises and problems, auto-grading, and assignable sets of questions that can be assigned to students by the click of mouse.
We express our appreciation to the following colleagues who contributed comments and suggestions that were valuable to us in the evolution of this text:

Nizar Bu Fakhreeddine, Department of Mathematics and Statistics, Notre Dame University
Zouk Mousbeh, Lebanon
Dr. Maged Iskander, Faculty of Business Administration, Economics and Political Science,
British University in Egypt
Dr. Fuad A. Kittaneh, Department of Mathematics, University of Jordan, Jordan
Haitham S. Solh, Department of Mathematics, American University in Dubai, UAE
Michael M. Zalzali, Department of Mathematics, UAE University, UAE

Many reviewers and contributors have provided valuable contributions and suggestions for previous editions of *Introductory Mathematical Analysis*. Many thanks to them for their insights, which have informed our work on this adaptation.

*Saadia Khouryibaba*
Introductory Mathematical Analysis
For Business, Economics, and the Life and Social Sciences
Arab World Edition
13 Integration

A nyone who runs a business knows the need for accurate cost estimates. When jobs are individually contracted, determining how much a job will cost is generally the first step in deciding how much to bid.

For example, a painter must determine how much paint a job will take. Since a gallon of paint will cover a certain number of square meters, the key is to determine the area of the surfaces to be painted. Normally, even this requires only simple arithmetic—walls and ceilings are rectangular, and so total area is a sum of products of base and height.

But not all area calculations are as simple. Suppose, for instance, that the bridge shown below must be sandblasted to remove accumulated soot. How would the contractor who charges for sandblasting by the square meter calculate the area of the vertical face on each side of the bridge?

The area could be estimated as perhaps three-quarters of the area of the trapezoid formed by points $A$, $B$, $C$, and $D$. But a more accurate calculation—which might be desirable if the bid were for dozens of bridges of the same dimensions (as along a stretch of railroad)—would require a more refined approach.

If the shape of the bridge’s arch can be described mathematically by a function, the contractor could use the method introduced in this chapter: integration. Integration has many applications, the simplest of which is finding areas of regions bounded by curves. Other applications include calculating the total deflection of a beam due to bending stress, calculating the distance traveled underwater by a submarine, and calculating the electricity bill for a company that consumes power at differing rates over the course of a month. Chapters 10–12 dealt with differential calculus. We differentiated a function and obtained another function, its derivative. Integral calculus is concerned with the reverse process: We are given the derivative of a function and must find the original function. The need for doing this arises in a natural way. For example, we might have a marginal-revenue function and want to find the revenue function from it. Integral calculus also involves a concept that allows us to take the limit of a special kind of sum as the number of terms in the sum becomes infinite. This is the real power of integral calculus! With such a notion, we can find the area of a region that cannot be found by any other convenient method.
Objective

To define the differential, interpret it geometrically, and use it in approximations. Also, to restate the reciprocal relationship between \( dx \) and \( dy \).

### 13.1 Differentials

We will soon give a reason for using the symbol \( dy/dx \) to denote the derivative of \( y \) with respect to \( x \). To do this, we introduce the notion of the differential of a function.

**Definition**

Let \( y = f(x) \) be a differentiable function of \( x \), and let \( \Delta x \) denote a change in \( x \), where \( \Delta x \) can be any real number. Then the differential of \( y \), denoted \( dy \) or \( d(f(x)) \), is given by

\[
    dy = f'(x) \Delta x
\]

Note that \( dy \) depends on two variables, namely, \( x \) and \( \Delta x \). In fact, \( dy \) is a function of two variables.

**EXAMPLE 1 Computing a Differential**

Find the differential of \( y = x^3 - 2x^2 + 3x - 4 \), and evaluate it when \( x = 1 \) and \( \Delta x = 0.04 \).

**Solution:** The differential is

\[
    dy = \frac{d}{dx}(x^3 - 2x^2 + 3x - 4) \Delta x = (3x^2 - 4x + 3) \Delta x
\]

When \( x = 1 \) and \( \Delta x = 0.04 \),

\[
    dy = [3(1)^2 - 4(1) + 3](0.04) = 0.08
\]

*Now Work Problem 1 <i>*</i>*

If \( y = x \), then \( dy = dx = \Delta x \). Hence, the differential of \( x \) is \( \Delta x \). We abbreviate \( d(x) \) by \( dx \). Thus, \( dx = \Delta x \). From now on, it will be our practice to write \( dx \) for \( \Delta x \) when finding a differential. For example,

\[
    d(x^2 + 5) = \frac{d}{dx}(x^2 + 5) dx = 2x \, dx
\]

Summarizing, we say that if \( y = f(x) \) defines a differentiable function of \( x \), then

\[
    dy = f'(x) \, dx
\]

where \( dx \) is any real number. Provided that \( dx \neq 0 \), we can divide both sides by \( dx \):

\[
    \frac{dy}{dx} = f'(x)
\]

That is, \( dy/dx \) can be viewed either as the quotient of two differentials, namely, \( dy \) divided by \( dx \), or as one symbol for the derivative of \( f \) at \( x \). It is for this reason that we introduced the symbol \( dy/dx \) to denote the derivative.

**EXAMPLE 2 Finding a Differential in Terms of \( dx \)**

a. If \( f(x) = \sqrt{x} \), then

\[
    d(\sqrt{x}) = \frac{d}{dx}(\sqrt{x}) \, dx = \frac{1}{2}x^{-1/2} \, dx = \frac{1}{2\sqrt{x}} \, dx
\]

b. If \( u = (x^2 + 3)^5 \), then \( du = 5(x^2 + 3)^4(2x) \, dx = 10(x^2 + 3)^4 \, dx \).

*Now Work Problem 3 <i>*</i>*
The differential can be interpreted geometrically. In Figure 13.1, the point 
\( P(x, f(x)) \) is on the curve \( y = f(x) \). Suppose \( x \) changes by \( dx \), a real number, to the
new value \( x + dx \). Then the new function value is \( f(x + dx) \), and the corresponding
point on the curve is \( Q(x + dx, f(x + dx)) \). Passing through \( P \) and \( Q \) are horizontal
and vertical lines, respectively, that intersect at \( S \). A line \( L \) tangent to the curve at \( P \)
intersects segment \( QS \) at \( R \), forming the right triangle \( PRS \). Observe that the graph of
\( f \) near \( P \) is approximated by the tangent line at \( P \). The slope of
\( L \) is \( f'(x) \) but it is also
given by \( \frac{SR}{PS} \) so that
\[
f'(x) = \frac{SR}{PS}
\]
Since \( dy = f'(x) \, dx \) and \( dx = PS \),
\[
dy = f'(x) \, dx = \frac{SR}{PS} \, PS = SR
\]
Thus, if \( dx \) is a change in \( x \) at \( P \), then \( dy \) is the corresponding vertical change along
the tangent line at \( P \). Note that for the same \( dx \), the vertical change along the curve
is \( \Delta y = SQ = f(x + dx) - f(x) \). Do not confuse \( \Delta y \) with \( dy \). However, from Figure 13.1, the following is apparent:

When \( dx \) is close to 0, \( dy \) is an approximation to \( \Delta y \). Therefore,
\[
\Delta y \approx dy
\]
This fact is useful in estimating \( \Delta y \), a change in \( y \), as Example 3 shows.

### Example 3 Using the Differential to Estimate a Change in a Quantity

A governmental health agency in the Middle East examined the records of a group of
individuals who were hospitalized with a particular illness. It was found that the total
proportion \( P \) that are discharged at the end of \( t \) days of hospitalization is given by
\[
P = P(t) = 1 - \left( \frac{300}{300 + t} \right)^3
\]
Use differentials to approximate the change in the proportion discharged if \( t \) changes
from 300 to 305.

**Solution:** The change in \( t \) from 300 to 305 is \( \Delta t = dt = 305 - 300 = 5 \). The change
in \( P \) is \( \Delta P = P(305) - P(300) \). We approximate \( \Delta P \) by \( dP \):
\[
\Delta P \approx dP = P'(t) \, dt = -3 \left( \frac{300}{300 + t} \right)^2 \left( -\frac{300}{(300 + t)^2} \right) \, dt = 3 \left( \frac{300}{300 + t} \right)^2 \, dt
\]
When \( t = 300 \) and \( dt = 5 \),
\[
dP = 3 \left( \frac{300^3}{600^4} \right) dt = \frac{15}{2^3 140} = \frac{1}{320} \approx 0.0031
\]
For a comparison, the true value of \( \Delta P \) is
\[
P(305) - P(300) = 0.87807 - 0.87500 = 0.00307
\]
(to five decimal places).

**Now Work Problem 11 ⊳**

We said that if \( y = f(x) \), then \( \Delta y \approx dy \) if \( dx \) is close to zero. Thus,
\[
\Delta y = f(x + dx) - f(x) \approx dy
\]
so that
\[
f(x + dx) \approx f(x) + dy
\]
This formula gives us a way of estimating a function value \( f(x + dx) \). For example, suppose we estimate \( \ln(1.06) \). Letting \( y = f(x) = \ln x \), we need to estimate \( f(1.06) \).
Since \( d(\ln x) = (1/x) \, dx \), we have, from Formula (1),
\[
f(x + dx) \approx f(x) + dy
\]
\[
\ln (x + dx) \approx \ln x + \frac{1}{x} \, dx
\]
We know the exact value of \( \ln 1 \), so we will let \( x = 1 \) and \( dx = 0.06 \). Then \( x + dx = 1.06 \), and \( dx \) is close to zero. Therefore,
\[
\ln (1 + 0.06) \approx \ln (1) + \frac{1}{1}(0.06)
\]
\[
\ln (1.06) \approx 0 + 0.06 = 0.06
\]
The true value of \( \ln(1.06) \) to five decimal places is 0.05827.

**EXAMPLE 4 Using the Differential to Estimate a Function Value**

A shoe manufacturer in Sudan established that the demand function for its sports shoes is given by
\[
p = f(q) = 20 - \sqrt{q}
\]
where \( p \) is the price per pair of shoes in dollars for \( q \) pairs. By using differentials, approximate the price when 99 pairs of shoes are demanded.

**Solution:** We want to approximate \( f(99) \). By Formula (1),
\[
f(q + dq) \approx f(q) + dp
\]
where
\[
dp = -\frac{1}{2\sqrt{q}} \, dq \quad \frac{dp}{dq} = -\frac{1}{2} q^{-1/2}
\]
We choose \( q = 100 \) and \( dq = -1 \) because \( q + dq = 99 \), \( dq \) is small, and it is easy to compute \( f(100) = 20 - \sqrt{100} = 10 \). We thus have
\[
f(99) = f[100 + (-1)] \approx f(100) - \frac{1}{2\sqrt{100}}(-1)
\]
\[
f(99) \approx 10 + 0.05 = 10.05
\]
Hence, the price per pair of shoes when 99 pairs are demanded is approximately $10.05.

**Now Work Problem 15 ⊳**

The equation \( y = x^3 + 4x + 5 \) defines \( y \) as a function of \( x \). We could write \( f(x) = x^3 + 4x + 5 \). However, the equation also defines \( x \) implicitly as a function of \( y \). In fact,
if we restrict the domain of $f$ to some set of real numbers $x$ so that $y = f(x)$ is a one-to-one function, then in principle we could solve for $x$ in terms of $y$ and get $x = f^{-1}(y)$. [Actually, no restriction of the domain is necessary here. Since $f'(x) = 3x^2 + 4 > 0$, for all $x$, we see that $f$ is strictly increasing on $(-\infty, \infty)$ and is thus one-to-one on $(-\infty, \infty).$] As we did in Section 11.2, we can look at the derivative of $x$ with respect to $y$, $dx/dy$ and we have seen that it is given by

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$ provided that $dy/dx \neq 0$$

Since $dx/dy$ can be considered a quotient of differentials, we now see that it is the reciprocal of the quotient of differentials $dy/dx$. Thus

$$\frac{dx}{dy} = \frac{1}{3x^2 + 4}$$

It is important to understand that it is not necessary to be able to solve $y = x^3 + 4x + 5$ for $x$ in terms of $y$, and the equation $\frac{dx}{dy} = \frac{1}{3x^2 + 4}$ holds for all $x$.

**EXAMPLE 5** Finding $dp/dq$ from $dq/dp$

Find $\frac{dp}{dq}$ if $q = \sqrt{2500 - p^2}$.

**Solution:**

**Strategy** There are a number of ways to find $dp/dq$. One approach is to solve the given equation for $p$ explicitly in terms of $q$ and then differentiate directly. Another approach to find $dp/dq$ is to use implicit differentiation. However, since $q$ is given explicitly as a function of $p$, we can easily find $dq/dp$ and then use the preceding reciprocal relation to find $dp/dq$. We will take this approach.

We have

$$\frac{dq}{dp} = \frac{1}{2}(2500 - p^2)^{-1/2}(-2p) = -\frac{p}{\sqrt{2500 - p^2}}$$

Hence,

$$\frac{dp}{dq} = -\frac{1}{\frac{dq}{dp}} = -\frac{\sqrt{2500 - p^2}}{p}$$

Now Work Problem 25 ⬤

**PROBLEMS 13.1**

In Problems 1–9, find the differential of the function in terms of $x$ and $dx$.

1. $y = ax + b$
2. $y = 2$
3. $f(x) = \sqrt{x^2 - 9}$
4. $f(x) = (4x^2 - 5x + 2)^3$
5. $u = \frac{1}{x^2}$
6. $u = \sqrt{x}$
7. $p = \ln(x^2 + 7)$
8. $p = e^{x^2 + 2x - 5}$
9. $y = \ln \sqrt{x^2 + 12}$

In Problems 10–14, find $\Delta y$ and $dy$ for the given values of $x$ and $dx$.

10. $y = 5x^2; x = -1, dx = -0.02$
11. $y = ax + b; \text{for any } x \text{ and any } dx$
12. $y = 2x^2 + 5x - 7; x = -2, dx = 0.1$
13. $y = (3x + 2)^2; x = -1, dx = -0.03$
14. $y = \sqrt{32 - x^2}; x = 4, dx = -0.05$ Round your answer to three decimal places.

15. Let $f(x) = \frac{x + 5}{x + 1}$.
   (a) Evaluate $f'(1)$.
   (b) Use differentials to estimate the value of $f(1.1)$.

In Problems 16–23, approximate each expression by using differentials.

16. $\sqrt{288}$ (Hint: $17^2 = 289.$) 17. $\sqrt{122}$
18. $\sqrt{5}$ 19. $\sqrt{16.3}$
20. In 0.97  
21. In 1.01  
22. $e^{0.001}$  
23. $e^{-0.002}$  

In Problems 24–29, find $dx/dy$ or $dp/dq$.

24. $y = 5x^2 + 3x + 2$  
25. $y = 2x - 1$  
26. $q = (p^2 + 5)^3$  
27. $q = \sqrt{p + 5}$  
28. $q = \frac{1}{p}$  
29. $q = e^{3-2p}$  

30. If $y = 7x^2 - 6x + 3$, find the value of $dx/dy$ when $x = 3$.  
31. If $y = \ln x^2$, find the value of $dx/dy$ when $x = 3$.  

In Problems 32 and 33, find the rate of change of $q$ with respect to $p$ for the indicated value of $q$.

32. $p = \frac{500}{q + 2}; q = 18$  
33. $p = 60 - \sqrt{2q}; q = 50$  

34. **Profit** Suppose that the profit (in dollars) of producing $q$ units of a product is 

\[ P = 397q - 2.3q^2 - 400 \]

Using differentials, find the approximate change in profit if the level of production changes from $q = 90$ to $q = 91$. Find the true change.

35. **Revenue** Given the revenue function 

\[ r = 250q + 45q^2 - q^3 \]

use differentials to find the approximate change in revenue if the number of units increases from $q = 40$ to $q = 41$. Find the true change.

36. **Demand** The demand equation for a product is 

\[ p = \frac{10}{\sqrt{q}} \]

Using differentials, approximate the price when 24 units are demanded.

37. **Demand** Given the demand function 

\[ p = \frac{200}{\sqrt{q} + 8} \]

use differentials to estimate the price per unit when 40 units are demanded.

38. If $y = f(x)$, then the proportional change in $y$ is defined to be $\Delta y/y$, which can be approximated with differentials by $dy/y$. Use this last form to approximate the proportional change in the cost function 

\[ c = f(q) = \frac{q^2}{2} + 5q + 300 \]

when $q = 10$ and $dq = 2$. Round your answer to one decimal place.

39. **Status/Income** Suppose that $S$ is a numerical value of status based on a person’s annual income $I$ (in thousands of dollars). For a certain population, suppose $S = 20\sqrt{I}$. Use differentials to approximate the change in $S$ if annual income decreases from $45,000$ to $44,500$.

40. **Biology** The volume of a spherical cell is given by 

\[ V = \frac{4}{3}\pi r^3 \]

where $r$ is the radius. Estimate the change in volume when the radius changes from $6.5 \times 10^{-4}$ cm to $6.6 \times 10^{-4}$ cm.

41. **Muscle Contraction** The equation 

\[ (P + a)(v + b) = k \]

is called the “fundamental equation of muscle contraction.” Here $P$ is the load imposed on the muscle, $v$ is the velocity of the shortening of the muscle fibers, and $a$, $b$, and $k$ are positive constants. Find $P$ in terms of $v$, and then use the differential to approximate the change in $P$ due to a small change in $v$.

42. **Profit** The demand equation for a monopolist’s product is 

\[ p = \frac{1}{2}q^2 - 66q + 7000 \]

and the average-cost function is 

\[ \bar{c} = 500 - q + \frac{80,000}{2q} \]

(a) Find the profit when 100 units are demanded.  
(b) Use differentials and the result of part (a) to estimate the profit when 101 units are demanded.

---

**Definition**

An antiderivative of a function \( f \) is a function \( F \) such that

\[
F'(x) = f(x)
\]

Equivalently, in differential notation,

\[
dF = f(x) \, dx
\]

For example, because the derivative of \( x^2 \) is \( 2x \), \( x^2 \) is an antiderivative of \( 2x \). However, it is not the only antiderivative of \( 2x \): Since

\[
\frac{d}{dx}(x^2 + 1) = 2x \quad \text{and} \quad \frac{d}{dx}(x^2 - 5) = 2x
\]

both \( x^2 + 1 \) and \( x^2 - 5 \) are also antiderivatives of \( 2x \). In fact, it is obvious that because the derivative of a constant is zero, \( x^2 + C \) is also an antiderivative of \( 2x \) for any constant \( C \). Thus, \( 2x \) has infinitely many antiderivatives. More importantly, all antiderivatives of \( 2x \) must be functions of the form \( x^2 + C \), because of the following fact:

Any two antiderivatives of a function differ only by a constant.

Since \( x^2 + C \) describes all antiderivatives of \( 2x \), we can refer to it as being the most general antiderivative of \( 2x \), denoted by \( \int 2x \, dx \), which is read “the indefinite integral of \( 2x \) with respect to \( x \).” Thus, we write

\[
\int 2x \, dx = x^2 + C
\]

The symbol \( \int \) is called the integral sign, \( 2x \) is the integrand, and \( C \) is the constant of integration. The \( dx \) is part of the integral notation and indicates the variable involved. Here \( x \) is the variable of integration.

More generally, the indefinite integral of any function \( f \) with respect to \( x \) is written \( \int f(x) \, dx \) and denotes the most general antiderivative of \( f \). Since all antiderivatives of \( f \) differ only by a constant, if \( F \) is any antiderivative of \( f \), then

\[
\int f(x) \, dx = F(x) + C, \quad \text{where } C \text{ is a constant}
\]

To integrate \( f \) means to find \( \int f(x) \, dx \). In summary,

\[
\int f(x) \, dx = F(x) + C \quad \text{if and only if} \quad F'(x) = f(x)
\]

Thus we have

\[
\frac{d}{dx} \left( \int f(x) \, dx \right) = f(x) \quad \text{and} \quad \int \frac{d}{dx} (F(x)) \, dx = F(x) + C
\]

which shows the extent to which differentiation and indefinite integration are inverse procedures.
EXAMPLE 6 Finding an Indefinite Integral

Find \( \int 5 \, dx \).

Solution:

Strategy First we must find (perhaps better words are guess at) a function whose derivative is 5. Then we add the constant of integration.

Since we know that the derivative of \( 5x \) is 5, \( 5x \) is an antiderivative of 5. Therefore,

\[
\int 5 \, dx = 5x + C
\]

Now Work Problem 1

Table 13.1 Elementary Integration Formulas

<table>
<thead>
<tr>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \int k , dx = kx + C )</td>
<td>( k ) is a constant</td>
</tr>
<tr>
<td>2. ( \int x^a , dx = \frac{x^{a+1}}{a+1} + C )</td>
<td>( a \neq -1 )</td>
</tr>
<tr>
<td>3. ( \int x^{-1} , dx = \int \frac{1}{x} , dx = \int \frac{dx}{x} = \ln x + C )</td>
<td>( x &gt; 0 )</td>
</tr>
<tr>
<td>4. ( \int e^x , dx = e^x + C )</td>
<td></td>
</tr>
<tr>
<td>5. ( \int kf(x) , dx = k \int f(x) , dx )</td>
<td>( k ) is a constant</td>
</tr>
<tr>
<td>6. ( \int (f(x) \pm g(x)) , dx = \int f(x) , dx \pm \int g(x) , dx )</td>
<td></td>
</tr>
</tbody>
</table>

Using differentiation formulas from Chapters 10 and 11, we have compiled a list of elementary integration formulas in Table 13.1. These formulas are easily verified. For example, Formula (2) is true because the derivative of \( \frac{x^{a+1}}{a+1} \) is \( x^a \) for \( a \neq -1 \). (We must have \( a \neq -1 \) because the denominator is 0 when \( a = -1 \).) Formula (2) states that the indefinite integral of a power of \( x \), other than \( x^{-1} \), is obtained by increasing the exponent of \( x \) by 1, dividing by the new exponent, and adding a constant of integration. The indefinite integral of \( x^{-1} \) will be discussed in Section 13.4.

To verify Formula (5), we must show that the derivative of \( k \int f(x) \, dx \) is \( kf(x) \). Since the derivative of \( k \int f(x) \, dx \) is simply \( k \) times the derivative of \( \int f(x) \, dx \), and the derivative of \( \int f(x) \, dx \) is \( f(x) \), Formula (5) is verified. The reader should verify the other formulas. Formula (6) can be extended to any number of terms.

EXAMPLE 7 Indefinite Integrals of a Constant and of a Power of \( x \)

a. Find \( \int 1 \, dx \).

Solution: By Formula (1) with \( k = 1 \)

\[
\int 1 \, dx = x + C
\]

Usually, we write \( \int 1 \, dx \) as \( \int dx \). Thus, \( \int dx = x + C \).
b. Find \( \int x^5 \, dx \).

**Solution:** By Formula (2) with \( n = 5 \),

\[
\int x^5 \, dx = \frac{x^{5+1}}{5+1} + C = \frac{x^6}{6} + C
\]

Now Work Problem 3

---

**EXAMPLE 8** Indefinite Integral of a Constant Times a Function

Find \( \int 7x \, dx \).

**Solution:** By Formula (5) with \( k = 7 \) and \( f(x) = x \),

\[
\int 7x \, dx = 7 \int x \, dx
\]

Since \( x \) is \( x^1 \), by Formula (2) we have

\[
\int x^1 \, dx = \frac{x^{1+1}}{1+1} + C_1 = \frac{x^2}{2} + C_1
\]

where \( C_1 \) is the constant of integration. Therefore,

\[
\int 7x \, dx = 7 \int x \, dx = 7 \left( \frac{x^2}{2} + C_1 \right) = \frac{7}{2}x^2 + 7C_1
\]

Since \( 7C_1 \) is just an arbitrary constant, we will replace it by \( C \) for simplicity. Thus,

\[
\int 7x \, dx = \frac{7}{2}x^2 + C
\]

It is not necessary to write all intermediate steps when integrating. More simply, we write

\[
\int 7x \, dx = \left( 7 \right) \frac{x^2}{2} + C = \frac{7}{2}x^2 + C
\]

Now Work Problem 5

---

**EXAMPLE 9** Indefinite Integral of a Constant Times a Function

Find \( \int -\frac{3}{5} e^x \, dx \).

**Solution:**

\[
\int -\frac{3}{5} e^x \, dx = -\frac{3}{5} \int e^x \, dx
\]

By Formula (4),

\[
= -\frac{3}{5} e^x + C
\]

Now Work Problem 21

---

**EXAMPLE 10** Finding Indefinite Integrals

a. Find \( \int \frac{1}{\sqrt{t}} \, dt \).

**Solution:** Here \( t \) is the variable of integration. We rewrite the integrand so that a basic formula can be used. Since \( 1/\sqrt{t} = t^{-1/2} \), applying Formula (2) gives

\[
\int \frac{1}{\sqrt{t}} \, dt = \int t^{-1/2} \, dt = \frac{t^{(-1/2)+1}}{-\frac{1}{2} + 1} + C = \frac{t^{1/2}}{\frac{1}{2}} + C = 2\sqrt{t} + C
\]
b. Find \( \int \frac{1}{6x^3} \, dx \).

Solution:

\[ \int \frac{1}{6x^3} \, dx = \frac{1}{6} \int x^{-3} \, dx = \left( \frac{1}{6} \right) \frac{x^{-3+1}}{-3+1} + C \]
\[ = -\frac{x^{-2}}{12} + C = -\frac{1}{12x^2} + C \]

Now Work Problem 9

EXAMPLE 11 Indefinite Integral of a Sum

Find \( \int (x^2 + 2x) \, dx \).

Solution: By Formula (6),

\[ \int (x^2 + 2x) \, dx = \int x^2 \, dx + \int 2x \, dx \]

Now,

\[ \int x^2 \, dx = \frac{x^{2+1}}{2+1} + C_1 = \frac{x^3}{3} + C_1 \]

and

\[ \int 2x \, dx = 2 \int x \, dx = \frac{2x^{1+1}}{1+1} + C_2 = x^2 + C_2 \]

Thus,

\[ \int (x^2 + 2x) \, dx = \frac{x^3}{3} + x^2 + C_1 + C_2 \]

For convenience, we will replace the constant \( C_1 + C_2 \) by \( C \). We then have

\[ \int (x^2 + 2x) \, dx = \frac{x^3}{3} + x^2 + C \]

Omitting intermediate steps, we simply integrate term by term and write

\[ \int (x^2 + 2x) \, dx = \frac{x^3}{3} + (2) \frac{x^2}{2} + C = \frac{x^3}{3} + x^2 + C \]

Now Work Problem 11

EXAMPLE 12 Indefinite Integral of a Sum and Difference

Find \( \int (2\sqrt{x} - 7x^3 + 10e^x - 1) \, dx \).

Solution:

\[ \int (2\sqrt{x} - 7x^3 + 10e^x - 1) \, dx \]

\[ = 2 \int x^{1/2} \, dx - 7 \int x^3 \, dx + 10 \int e^x \, dx - \int 1 \, dx \quad \text{Formulas (5) and (6)} \]

\[ = (2) \frac{x^{3/2}}{3/2} - (7) \frac{x^4}{4} + 10e^x - x + C \quad \text{Formulas (1), (2), and (4)} \]

\[ = \frac{10}{9} x^{9/5} - \frac{7}{4} x^4 + 10e^x - x + C \]

Now Work Problem 15
Sometimes, in order to apply the basic integration formulas, it is necessary first to perform algebraic manipulations on the integrand, as Example 13 shows.

**EXAMPLE 13** Using Algebraic Manipulation to Find an Indefinite Integral

Find \( \int y^2 \left( y + \frac{2}{3} \right) \, dy \).

**Solution:** The integrand does not fit a familiar integration form. However, by multiplying the integrand we get

\[
\int y^2 \left( y + \frac{2}{3} \right) \, dy = \int \left( y^3 + \frac{2}{3} y^2 \right) \, dy
\]

\[
= \frac{y^4}{4} + \left( \frac{2}{3} \right) \frac{y^3}{3} + C = \frac{y^4}{4} + \frac{2y^3}{9} + C
\]

Now Work Problem 39 ⊳

**EXAMPLE 14** Using Algebraic Manipulation to Find an Indefinite Integral

a. Find \( \int \frac{(2x - 1)(x + 3)}{6} \, dx \).

**Solution:** By factoring out the constant \( \frac{1}{6} \) and multiplying the binomials, we get

\[
\int \frac{(2x - 1)(x + 3)}{6} \, dx = \frac{1}{6} \int (2x^2 + 5x - 3) \, dx
\]

\[
= \frac{1}{6} \left( \frac{2}{3} x^3 + \frac{5}{2} x^2 - 3x \right) + C
\]

\[
= \frac{x^3}{9} + \frac{5x^2}{12} - \frac{x}{2} + C
\]

Another algebraic approach to part (b) is

\[
\int \frac{x^3 - 1}{x^2} \, dx = \int \left( x - \frac{1}{x^2} \right) \, dx
\]

and so on.

b. Find \( \int \frac{x^3 - 1}{x^2} \, dx \).

**Solution:** We can break up the integrand into fractions by dividing each term in the numerator by the denominator:

\[
\int \frac{x^3 - 1}{x^2} \, dx = \int \left( \frac{x^3}{x^2} - \frac{1}{x^2} \right) \, dx = \int \left( x - \frac{1}{x^2} \right) \, dx
\]

\[
= \frac{x^2}{2} - \frac{x^{-1}}{-1} + C = \frac{x^2}{2} + \frac{1}{x} + C
\]

Now Work Problem 47 ⊳

**PROBLEMS 13.2**

In Problems 1–50, find the indefinite integrals.

1. \( \int 7 \, dx \)
2. \( \int \frac{1}{x} \, dx \)
3. \( \int x^8 \, dx \)
4. \( \int 5x^3 \, dx \)
5. \( \int 5x^{-3} \, dx \)
6. \( \int \frac{x^3}{3} \, dx \)
7. \( \int \frac{5}{x^7} \, dx \)
8. \( \int \frac{7}{2x^{9/4}} \, dx \)
9. \( \int \frac{1}{t^{3/4}} \, dt \)
10. \( \int (7t^3 + 4t^2 + 1) \, dt \)
11. \( \int (4 + t) \, dt \)
12. \( \int (y^5 - 5y) \, dy \)
13. \( \int (5 - 2w - 6w^3) \, dw \)
14. \( \int (1 + t^2 + t^4 + t^6) \, dt \)
15. \( \int (3t^2 - 4t + 5) \, dt \)
16. \( \int (\sqrt{2} + e) \, dx \)
17. \( \int (5 - 2^{-1}) \, dx \)  
18. \( \int \left( \frac{x}{7} - \frac{3}{4}x^4 \right) \, dx \)  
19. \( \int \left( \frac{2x^2}{7} - \frac{8}{3}x^5 \right) \, dx \)  
20. \( \int (e^x + 3x^2 + 2x) \, dx \)  
21. \( \int \pi e^x \, dx \)  
22. \( \int (x^3 - 9x^6 + 3x^{-4} + x^{-3}) \, dx \)  
23. \( \int (0.7y^3 + 10 + 2y^{-3}) \, dy \)  
24. \( \int \frac{-2\sqrt{x}}{3} \, dx \)  
25. \( \int dz \)  
26. \( \int \frac{5}{3\sqrt{x^2}} \, dx \)  
27. \( \int \frac{-4}{(3x)^3} \, dx \)  
28. \( \int \left( \frac{x^3}{3} - \frac{3}{x^3} \right) \, dx \)  
29. \( \int \left( \frac{1}{2x^3} - \frac{1}{x^4} \right) \, dx \)  
30. \( \int \left( \frac{3u^2}{2} - \frac{2}{3w^2} \right) \, dw \)  
31. \( \int 7e^{-3x} \, ds \)  
32. \( \int \frac{3u - 4}{5} \, du \)  
33. \( \int \frac{1}{12} \left( \frac{1}{3}e^t \right) \, dx \)  
34. \( \int (u^e + e^u) \, du \)  
35. \( \int \left( \frac{3}{\sqrt{x}} - 12\sqrt{x} \right) \, dx \)  
36. \( \int 0 \, dt \)  
37. \( \int \left( -\frac{\sqrt{2x^2}}{5} - \frac{7}{2\sqrt{x}} + 6x \right) \, dx \)  
38. \( \int \left( \frac{\sqrt{a} + 1}{\sqrt{a}} \right) \, du \)  
39. \( \int (x^2 + 5)(x - 3) \, dx \)  
40. \( \int x^3(x^2 + 5x + 2) \, dx \)  
41. \( \int \sqrt{3}(x + 3) \, dx \)  
42. \( \int (z + 2)^3 \, dz \)  
43. \( \int (3u + 2)^3 \, du \)  
44. \( \int \left( \frac{4}{\sqrt{x}} - 1 \right)^2 \, dx \)  
45. \( \int \sqrt{3}(x^4 + 4x^2 - 5) \, dx \)  
46. \( \int (6e^x + u^3)(\sqrt{a} + 1) \, du \)  
47. \( \int \frac{x^4 + 10}{2e^2} \, dz \)  
48. \( \int \frac{x^4 - 5x^2 + 2x}{5x^2} \, dx \)  
49. \( \int e^x + e^{2x} \, dx \)  
50. \( \int \left( \frac{x^2 + 1}{x} \right) \, dx \)  
51. If \( F(x) \) and \( G(x) \) are such that \( F'(x) = G(x) \), is it true that \( F(x) - G(x) \) must be zero?  
52. (a) Find a function \( F \) such that \( \int F(x) \, dx = xe^x + C \).  
(b) Is there only one function \( F \) satisfying the equation given in part (a), or are there many such functions?  
53. Find \( \int \frac{d}{dx} \left( \frac{1}{\sqrt{x^2 + 1}} \right) \, dx \).  

### 13.3 Integration with Initial Conditions

To find a particular antiderivative of a function that satisfies certain conditions. This involves evaluating constants of integration.

If we know the rate of change, \( f' \), of the function \( f \), then the function \( f \) itself is an antiderivative of \( f' \) (since the derivative of \( f \) is \( f' \)). Of course, there are many antiderivatives of \( f' \), and the most general one is denoted by the indefinite integral. For example, if

\[
f'(x) = 2x
\]

then

\[
f(x) = \int f'(x) \, dx = \int 2x \, dx = x^2 + C. \tag{1}
\]

That is, any function of the form \( f(x) = x^2 + C \) has its derivative equal to \( 2x \). Because of the constant of integration, notice that we do not know \( f(x) \) specifically. However, if \( f \) must assume a certain function value for a particular value of \( x \), then we can determine the value of \( C \) and thus determine \( f(x) \) specifically. For instance, if \( f(1) = 4 \), then, from Equation (1),

\[
f(1) = 1^2 + C \\
4 = 1 + C \\
\]

\[
C = 3
\]

Thus,

\[
f(x) = x^2 + 3
\]

That is, we now know the particular function \( f(x) \) for which \( f'(x) = 2x \) and \( f(1) = 4 \). The condition \( f(1) = 4 \), which gives a function value of \( f \) for a specific value of \( x \), is called an initial condition.
**EXAMPLE 15** Initial-Condition Problem

Suppose that the marginal profit of a plastics factory in Qatar is given by the function

\[ P'(x) = \frac{x^2}{25} - 3x + 150 \]

where \( x \) is the number (in thousands) of items produced and \( P \) represents the profit in thousands of dollars. Find the profit function, assuming that selling no items results in a loss of $400,000.

**Solution:** The profit function is

\[ P(x) = \int \frac{x^2}{25} - 3x + 150 \, dx = \int \frac{x^2}{25} \, dx - 3 \int x \, dx + 150 \int 1 \, dx \]

\[ = \frac{1}{25} \int x^2 \, dx - \frac{3}{2} x^2 + 150x + C \]

\[ = \frac{x^3}{75} - \frac{3x^2}{2} + 150x + C \quad (2) \]

We determine the value of \( C \) by using the initial condition: substitute \( x = 0 \) and \( P(0) = -400 \) into Equation (2) to get

\[ 0 = \frac{0^3}{75} - \frac{3(0)^2}{2} + 150(0) + C = -400 \]

Hence,

\[ C = -400 \]

Hence,

\[ P(x) = \frac{x^3}{75} - \frac{3x^2}{2} + 150x - 400 \quad (3) \]

**Now Work Problem 1**

**EXAMPLE 16** Initial-Condition Problem Involving \( y'' \)

Given that \( y'' = x^2 - 6 \), \( y'(0) = 2 \), and \( y(1) = -1 \), find \( y \).

**Solution:**

**Strategy** To go from \( y'' \) to \( y \), two integrations are needed: the first to take us from \( y'' \) to \( y' \) and the other to take us from \( y' \) to \( y \). Hence, there will be two constants of integration, which we will denote by \( C_1 \) and \( C_2 \).

Since \( y'' = \frac{d}{dx}(y') = x^2 - 6 \), \( y' \) is an antiderivative of \( x^2 - 6 \). Thus,

\[ y' = \int (x^2 - 6) \, dx = \frac{x^3}{3} - 6x + C_1 \quad (4) \]

Now, \( y'(0) = 2 \) means that \( y' = 2 \) when \( x = 0 \); therefore, from Equation (4), we have

\[ 2 = \frac{0^3}{3} - 6(0) + C_1 \]

Hence, \( C_1 = 2 \), so

\[ y' = \frac{x^3}{3} - 6x + 2 \]
By integration, we can find $y$:

\[
y = \int \left( \frac{x^3}{3} - 6x + 2 \right) \, dx
\]

\[
= \left( \frac{1}{3} \right) \frac{x^4}{4} - (6) \frac{x^2}{2} + 2x + C_2
\]

so

\[
y = \frac{x^4}{12} - 3x^2 + 2x + C_2 \tag{5}
\]

Now, since $y = -1$ when $x = 1$, we have, from Equation (5),

\[
-1 = \frac{1}{12} - 3(1)^2 + 2(1) + C_2
\]

Thus, $C_2 = -\frac{1}{12}$, so

\[
y = \frac{x^4}{12} - 3x^2 + 2x - \frac{1}{12}
\]

Integration with initial conditions is applicable to many applied situations, as the next three examples illustrate.

**EXAMPLE 17 Income and Education**

Suppose that for a particular Arab group, sociologists studied the current average yearly income $y$ (in dollars) that a person can expect to receive with $x$ years of education before seeking regular employment. They estimated that the rate at which income changes with respect to education is given by

\[
\frac{dy}{dx} = 100x^{3/2} \quad 4 \leq x \leq 16
\]

where $y = 28,720$ when $x = 9$. Find $y$.

**Solution:** Here $y$ is an antiderivative of $100x^{3/2}$. Thus,

\[
y = \int 100x^{3/2} \, dx = 100 \int x^{3/2} \, dx
\]

\[
= (100) \frac{x^{5/2}}{\frac{5}{2}} + C
\]

\[
y = 40x^{5/2} + C \tag{6}
\]

The initial condition is that $y = 28,720$ when $x = 9$. By putting these values into Equation (6), we can determine the value of $C$:

\[
28,720 = 40(9)^{5/2} + C
\]

\[
= 40(243) + C
\]

\[
28,720 = 9720 + C
\]

Therefore, $C = 19,000$, and

\[
y = 40x^{5/2} + 19,000
\]

Now Work Problem 17 <↓
**EXAMPLE 18** Finding Revenue from Marginal Average Revenue

Suppose that the marginal average revenue in dollars of Ali Baba Museum resulting from the sale of \( x \) tickets is given by

\[
\bar{R}(x) = 1 + \frac{1}{x}
\]

If the average revenue from the sale of 20 tickets is $25, what is the revenue when 50 tickets are sold?

**Solution:** To find the revenue function, we first find the average revenue. We have

\[
\bar{R}(x) = \int \left( 1 + \frac{1}{x} \right) dx = x + \ln |x| + C
\]

To find \( C \), we use the initial condition \( \bar{R}(20) = 25 \). This gives

\[
\bar{R}(20) = 20 + \ln(20) + C = 25
\]

\[
C = 5 - \ln(20) \approx 2
\]

Therefore \( \bar{R}(x) = x + \ln(x) + 2 \) and hence \( R(x) = x(x + \ln(x) + 2) \). So the revenue from the sale of 50 tickets is

\[
R(50) = 50(50 + \ln(50) + 2) \approx 2796 \text{ dollars}
\]

**EXAMPLE 19** Finding the Demand Function from Marginal Revenue

If the marginal-revenue function for a manufacturer’s product is

\[
\frac{dr}{dq} = 2000 - 20q - 3q^2
\]

find the demand function.

**Solution:**

**Strategy** By integrating \( \frac{dr}{dq} \) and using an initial condition, we can find the revenue function \( r \). But revenue is also given by the general relationship \( r = pq \), where \( p \) is the price per unit. Thus, \( p = r/q \). Replacing \( r \) in this equation by the revenue function yields the demand function.

Since \( dr/dq \) is the derivative of total revenue \( r \),

\[
r = \int (2000 - 20q - 3q^2) dq
\]

\[
= 2000q - (20)q^2/2 - (3)q^3/3 + C
\]

so that

\[
r = 2000q - 10q^2 - q^3 + C
\]

(7)

We assume that **when no units are sold, there is no revenue**; that is, \( r = 0 \) when \( q = 0 \). This is our initial condition. Putting these values into Equation (7) gives

\[
0 = 2000(0) - 10(0)^2 - 0^3 + C
\]

Hence, \( C = 0 \), and

\[
r = 2000q - 10q^2 - q^3
\]
To find the demand function, we use the fact that \( p = r/q \) and substitute for \( r \):

\[
p = \frac{r}{q} = \frac{2000q - 10q^2 - q^3}{q}
\]

\[
p = 2000 - 10q - q^2
\]

Now Work Problem 11

**Example 20** Finding Cost from Marginal Cost

Suppose that Al Hallab Restaurant’s fixed costs per week are $4000. (Fixed costs are costs, such as rent and insurance, that remain constant at all levels of production during a given time period.) If the marginal-cost function is

\[
\frac{dc}{dq} = 0.000001(0.002q^2 - 25q) + 0.2
\]

where \( c \) is the total cost (in dollars) of producing \( q \) meals per week, find the cost of producing 1000 meals in 1 week.

**Solution:** Since \( dc/dq \) is the derivative of the total cost \( c \),

\[
c(q) = \int [0.000001(0.002q^2 - 25q) + 0.2] \, dq
\]

\[
c(q) = 0.000001 \int (0.002q^2 - 25q) \, dq + \int 0.2 \, dq
\]

\[
c(q) = 0.000001 \left( \frac{0.002q^3}{3} - \frac{25q^2}{2} \right) + 0.2q + C
\]

Fixed costs are constant regardless of output. Therefore, when \( q = 0 \), total cost is equal to fixed cost. Putting \( c(0) = 4000 \) in the last equation, we find that \( C = 4000 \), so

\[
c(q) = 0.000001 \left( \frac{0.002q^3}{3} - \frac{25q^2}{2} \right) + 0.2q + 4000 \tag{8}
\]

From Equation (8), we have \( c(1000) = 4188.17 \). Thus, the total cost for producing 1000 meals in 1 week is $4188.17.

Now Work Problem 15

**Problems 13.3**

In Problems 1 and 2, find \( y \) subject to the given conditions.
1. \( dy/dx = 3x - 4; \quad y(-1) = \frac{13}{2} \)
2. \( dy/dx = x^2 - x; \quad y(3) = \frac{19}{2} \)

In Problems 3, if \( y \) satisfies the given conditions, find \( y(x) \) for the given value of \( x \).
3. \( y' = \frac{9}{8\sqrt{x}}; \quad y(16) = 10; \quad x = 9 \)

In Problems 4–7, find \( y \) subject to the given conditions.
4. \( y'' = x + 1; \quad y'(0) = 0, \quad y(0) = 5 \)
5. \( y'' = -3x^2 + 4x; \quad y'(1) = 2, \quad y(1) = 3 \)
6. \( y'' = 2x; \quad y'(1) = -3, \quad y(3) = 10, \quad y(0) = 13 \)
7. \( y'' = 2e^{-x} + 3; \quad y'(0) = 7, \quad y(0) = 5, \quad y(1) = 1 \)

In Problems 8–11, \( dr/dq \) is a marginal-revenue function. Find the demand function.
8. \( dr/dq = 0.7 \)
9. \( dr/dq = 10 - \frac{1}{16}q \)
10. \( dr/dq = 5000 - 3(2q + 2q^3) \)
11. \( dr/dq = 275 - q - 0.3q^2 \)
In Problems 12–15, \( dc/dq \) is a marginal-cost function and fixed costs are indicated in braces. For Problems 12 and 13, find the total-cost function. For Problems 14 and 15, find the total cost for the indicated value of \( q \).

12. \( dc/dq = 2.47; \quad \{159\} \)

13. \( dc/dq = 2q + 75; \quad \{2000\} \)

14. \( dc/dq = 0.000204q^2 - 0.046q + 6; \quad \{15,000\}; \quad q = 200 \)

15. \( dc/dq = 0.08q^2 - 1.6q + 6.5; \quad \{8000\}; \quad q = 25 \)

16. **Winter Moth** A study of the winter moth was made in Nova Scotia, Canada.\(^2\) The prepupae of the moth fall onto the ground from host trees. It was found that the (approximate) rate at which prepupal density \( y \) (the number of prepupae per square foot of soil) changes with respect to distance \( x \) (in feet) from the base of a host tree is

\[
\frac{dy}{dx} = -1.5 - x \quad 1 \leq x \leq 9
\]

If \( y = 59.6 \) when \( x = 1 \), find \( y \).

17. **Diet for Rats** A group of biologists studied the nutritional effects on rats that were fed a diet containing 10% protein.\(^3\) The protein consisted of yeast and corn flour.

Over a period of time, the group found that the (approximate) rate of change of the average weight gain \( G \) (in grams) of a rat with respect to the percentage \( P \) of yeast in the protein mix was

\[
\frac{dG}{dP} = -\frac{P}{25} + 2 \quad 0 \leq P \leq 100
\]

If \( G = 38 \) when \( P = 10 \), find \( G \).

18. **Fluid Flow** In the study of the flow of fluid in a tube of constant radius \( R \), such as blood flow in portions of the body, one can think of the tube as consisting of concentric tubes of radius \( r \), where \( 0 \leq r \leq R \). The velocity \( v \) of the fluid is a function of \( r \) and is given by\(^4\)

\[
v = \int \frac{(P_1 - P_2)r}{2\eta} \, dr
\]

where \( P_1 \) and \( P_2 \) are pressures at the ends of the tube, \( \eta \) (a Greek letter read “eta”) is fluid viscosity, and \( l \) is the length of the tube. If \( v = 0 \) when \( r = R \), show that

\[
v = \frac{(P_1 - P_2)(R^2 - r^2)}{4\eta l}
\]

19. **Average Cost** Amran manufactures jeans and has determined that the marginal-cost function is

\[
\frac{dc}{dq} = 0.003q^2 - 0.4q + 40
\]

where \( q \) is the number of pairs of jeans produced. If marginal cost is $27.50 when \( q = 50 \) and fixed costs are $5000, what is the average cost of producing 100 pairs of jeans?

20. If \( f''(x) = 30x^4 + 12x \) and \( f'(1) = 10 \), evaluate

\[
f(965.335245) - f(-965.335245)
\]

---


Thus,
\[ \int (u(x))^a \cdot u'(x) \, dx = \frac{(u(x))^{a+1}}{a+1} + C \quad a \neq -1 \]

We call this the **power rule for integration**. Note that \( u'(x) \, dx \) is the differential of \( u \), namely \( du \). In mathematical shorthand, we can replace \( u(x) \) by \( u \) and \( u'(x) \, dx \) by \( du \):

### Power Rule for Integration

If \( u \) is differentiable, then
\[ \int u^a \, du = \frac{u^{a+1}}{a+1} + C \quad \text{if} \quad a \neq -1 \quad (1) \]

It is important to appreciate the difference between the power rule for integration and the formula for \( \int x^a \, dx \). In the power rule, \( u \) represents a function, whereas in \( \int x^a \, dx \), \( x \) is a variable.

#### EXAMPLE 21 Applying the Power Rule for Integration

**a.** Find \( \int (x + 1)^{20} \, dx \).

**Solution:** Since the integrand is a power of the function \( x + 1 \), we will set \( u = x + 1 \). Then \( du = dx \), and \( \int (x + 1)^{20} \, dx \) has the form \( \int u^{20} \, du \). By the power rule for integration,
\[ \int (x + 1)^{20} \, dx = \int u^{20} \, du = \frac{u^{21}}{21} + C = \frac{(x + 1)^{21}}{21} + C \]

Note that we give our answer not in terms of \( u \), but explicitly in terms of \( x \).

**b.** Find \( \int 3x^2(x^3 + 7)^3 \, dx \).

**Solution:** We observe that the integrand contains a power of the function \( x^3 + 7 \). Let \( u = x^3 + 7 \). Then \( du = 3x^2 \, dx \). Fortunately, \( 3x^2 \) appears as a factor in the integrand and we have
\[ \int 3x^2(x^3 + 7)^3 \, dx = \int (x^3 + 7)^3[3x^2 \, dx] = \int u^3 \, du = \frac{u^4}{4} + C = \frac{(x^3 + 7)^4}{4} + C \]

After integrating, you may wonder what happened to \( 3x^2 \). We note again that \( du = 3x^2 \, dx \).

In order to apply the power rule for integration, sometimes an adjustment must be made to obtain \( du \) in the integrand, as Example 22 illustrates.

#### EXAMPLE 22 Adjusting for \( du \)

**Find** \( \int x\sqrt{x^2 + 5} \, dx \).

**Solution:** We can write this as \( \int x(x^2 + 5)^{1/2} \, dx \). Notice that the integrand contains a power of the function \( x^2 + 5 \). If \( u = x^2 + 5 \), then \( du = 2x \, dx \). Since the constant factor 2 in \( du \) does not appear in the integrand, this integral does not have the
form \( \int u^a \, du \). However, from \( du = 2x \, dx \) we can write \( x \, dx = \frac{du}{2} \) so that the integral becomes

\[
\int x(x^2 + 5)^{1/2} \, dx = \int (x^2 + 5)^{1/2} \, x \, dx = \int u^{1/2} \, \frac{du}{2}
\]

Moving the constant factor \( \frac{1}{2} \) in front of the integral sign, we have

\[
\int x(x^2 + 5)^{1/2} \, dx = \frac{1}{2} \int u^{1/2} \, du = \frac{1}{2} \left( \frac{u^{3/2}}{3/2} \right) + C = \frac{1}{3} u^{3/2} + C
\]

which in terms of \( x \) (as is required) gives

\[
\int x\sqrt{x^2 + 5} \, dx = \frac{(x^2 + 5)^{3/2}}{3} + C
\]

**Now Work Problem 15 ▽**

In Example 22, the integrand \( x\sqrt{x^2 + 5} \) missed being of the form \( (u(x))^{1/2} u'(x) \) by the constant factor of 2. In general, if we have

\[
\int (u(x))^{n} \frac{u'(x)}{k} \, dx,
\]

for \( k \) a nonzero constant, then we can write

\[
\int (u(x))^{n} \frac{u'(x)}{k} \, dx = \int \frac{u^n}{k} \, du = \frac{1}{k} \int u^n \, du
\]

to simplify the integral, but such adjustments of the integrand are not possible for variable factors.

When using the form \( \int u^n \, du \), do not neglect \( du \). For example,

\[
\int (4x + 1)^2 \, dx \neq \frac{(4x + 1)^3}{3} + C
\]

The correct way to do this problem is as follows. Let \( u = 4x + 1 \), from which it follows that \( du = 4 \, dx \). Thus \( dx = \frac{du}{4} \) and

\[
\int (4x + 1)^2 \, dx = \int u^2 \left[ \frac{du}{4} \right] = \frac{1}{4} \int u^2 \, du = \frac{1}{4} \cdot \frac{u^3}{3} + C = \frac{(4x + 1)^3}{12} + C
\]

**EXAMPLE 23**  **Applying the Power Rule for Integration**

**a.** Find \( \int \sqrt[3]{6y} \, dy \).

**Solution:** The integrand is \((6y)^{1/3}\), a power of a function. However, in this case the obvious substitution \( u = 6y \) can be avoided. More simply, we have

\[
\int \sqrt[3]{6y} \, dy = \int 6^{1/3} y^{1/3} \, dy = 6^{1/3} \int y^{1/3} \, dy = 6^{1/3} \frac{y^{4/3}}{4/3} + C = \frac{3\sqrt[3]{6}}{4} y^{4/3} + C
\]

**b.** Find \( \int \frac{2x^3 + 3x}{(x^4 + 3x^2 + 7)^4} \, dx \).

**Solution:** We can write this as \( \int (x^4 + 3x^2 + 7)^{-4} (2x^3 + 3x) \, dx \). Let us try to use the power rule for integration. If \( u = x^4 + 3x^2 + 7 \), then \( du = (4x^3 + 6x) \, dx \), which is two times the quantity \( (2x^3 + 3x) \, dx \) in the integral. Thus \( (2x^3 + 3x) \, dx = \frac{du}{2} \) and we again illustrate the adjustment technique:

\[
\int (x^4 + 3x^2 + 7)^{-4} (2x^3 + 3x) \, dx = \int u^{-4} \left[ \frac{du}{2} \right] = \frac{1}{2} \int u^{-4} \, du
\]

\[
= \frac{1}{2} \cdot \frac{u^{-3}}{-3} + C = -\frac{1}{6u^3} + C = -\frac{1}{6(x^4 + 3x^2 + 7)^3} + C
\]

**Now Work Problem 5 ▽**
In using the power rule for integration, take care when making a choice for \( u \). In Example 23(b), letting \( u = 2x^3 + 3x \) does not lead very far. At times it may be necessary to try many different choices. Sometimes a wrong choice will provide a hint as to what does work. **Skill at integration comes only after many hours of practice and conscientious study.**

**EXAMPLE 24** An Integral to Which the Power Rule Does Not Apply

Find \( \int 4x^2(x^4 + 1)^2 \, dx \).

**Solution:** If we set \( u = x^4 + 1 \), then \( du = 4x^3 \, dx \). To get \( du \) in the integral, we need an additional factor of the variable \( x \). However, we can adjust only for constant factors. Thus, we cannot use the power rule. Instead, to find the integral, we will first expand \( (x^4 + 1)^2 \):

\[
\int 4x^2(x^4 + 1)^2 \, dx = 4 \int x^2(x^8 + 2x^4 + 1) \, dx
\]

\[
= 4 \int (x^{10} + 2x^6 + x^2) \, dx
\]

\[
= 4 \left( \frac{x^{11}}{11} + \frac{2x^7}{7} + \frac{x^3}{3} \right) + C
\]

**Now Work Problem 65 \( \triangleright \)**

**Integrating Natural Exponential Functions**

We now turn our attention to integrating exponential functions. If \( u \) is a differentiable function of \( x \), then

\[
\frac{d}{dx}(e^u) = e^u \frac{du}{dx}
\]

Corresponding to this differentiation formula is the integration formula

\[
\int e^u \, du = e^u + C
\]

But \( \frac{du}{dx} \) is the differential of \( u \), namely, \( du \). Thus,

\[
\int e^u \, du = e^u + C \tag{2}
\]

**EXAMPLE 25** Integrals Involving Exponential Functions

a. Find \( \int 2xe^{x^2} \, dx \).

**Solution:** Let \( u = x^2 \). Then \( du = 2x \, dx \), and, by Equation (2),

\[
\int 2xe^{x^2} \, dx = \int e^u \, [2x \, dx] = \int e^u \, du
\]

\[
= e^u + C = e^{x^2} + C
\]
b. Find \( \int (x^2 + 1)e^{x^3 + 3x} \, dx \).

Solution: If \( u = x^3 + 3x \), then \( du = (3x^2 + 3) \, dx = 3(x^2 + 1) \, dx \). If the integrand contained a factor of 3, the integral would have the form \( \int e^u \, du \). Thus, we write

\[
\int (x^2 + 1)e^{x^3 + 3x} \, dx = \int e^{x^3 + 3x}[(x^2 + 1) \, dx] = \frac{1}{3} \int e^u \, du = \frac{1}{3} e^u + C
\]

\[
= \frac{1}{3} e^{x^3 + 3x} + C
\]

where in the second step we replaced \((x^2 + 1) \, dx\) by \(\frac{1}{3} \, du\) but wrote \(\frac{1}{3}\) outside the integral.

Now Work Problem 39

Integrals Involving Logarithmic Functions

As we know, the power-rule formula \( \int u^a \, du = u^{a+1}/(a+1) + C \) does not apply when \( a = -1 \). To handle that situation, namely, \( \int u^{-1} \, du = \int \frac{1}{u} \, du \), we first recall from Section 11.1 that

\[
\frac{d}{dx}(\ln |u|) = \frac{1}{u} \frac{du}{dx} \quad \text{for } u \neq 0
\]

which gives us the integration formula

\[
\int \frac{1}{u} \, du = \ln |u| + C \quad \text{for } u \neq 0
\] (3)

In particular, if \( u = x \), then \( du = dx \), and

\[
\int \frac{1}{x} \, dx = \ln |x| + C \quad \text{for } x \neq 0
\] (4)

**EXAMPLE 26** Integals Involving \( \frac{1}{u} \, du \)

a. Find \( \int \frac{7}{x} \, dx \).

Solution: From Equation (4),

\[
\int \frac{7}{x} \, dx = 7 \int \frac{1}{x} \, dx = 7 \ln |x| + C
\]

Using properties of logarithms, we can write this answer another way:

\[
\int \frac{7}{x} \, dx = \ln |x^7| + C
\]

b. Find \( \int \frac{2x}{x^2 + 5} \, dx \).

Solution: Let \( u = x^2 + 5 \). Then \( du = 2x \, dx \). From Equation (3),

\[
\int \frac{2x}{x^2 + 5} \, dx = \int \frac{1}{u} [2x \, dx] = \int \frac{1}{u} \, du
\]

\[
= \ln |u| + C = \ln |x^2 + 5| + C
\]

Since \( x^2 + 5 \) is always positive, we can omit the absolute-value bars:

\[
\int \frac{2x}{x^2 + 5} \, dx = \ln (x^2 + 5) + C
\]

Now Work Problem 29

**APPLY IT**

11. If the rate of vocabulary memorization of the average student in a foreign language is given by \( \frac{dv}{dt} = \frac{35}{t + 1} \), where \( v \) is the number of vocabulary words memorized in \( t \) hours of study, find the general form of \( v(t) \).
EXAMPLE 27  An Integral Involving $\frac{1}{u} du$

The manager of Al Madina Superstore determines that the price of a bottle of mineral water is changing at the rate of

$$\frac{dp}{dq} = -\frac{q}{2q^2 + 5}$$

where $q$ is the number (in hundreds) of bottles demanded by customers at a price $p$. If the quantity demanded is 150 when the price is $1.50, at what price would no bottles be sold?

Solution: The price function is

$$p(q) = \int \frac{-q}{2q^2 + 5} dq$$

Let $u = 2q^2 + 5$; then $du = 4q \, dq$, so that $q \, du = \frac{du}{4}$. Hence

$$p(q) = \int \frac{-q}{2q^2 + 5}dq = \int \frac{du}{4} = -\frac{1}{4} \int \frac{1}{u} du = -\frac{1}{4} \ln |u| + C$$

Rewriting $u$ in terms of $q$, we get

$$p(q) = -\frac{1}{4} \ln (2q^2 + 5) + C$$

To find the value of $C$, we use the fact that $p(150) = 1.5$. Thus,

$$p(150) = -\frac{1}{4} \ln (2(150)^2 + 5) + C = 1.5$$

$$C = 4.18$$

So $p(q) = -\frac{1}{4} \ln (2q^2 + 5) + 4.18$, and the price at which no bottles of water would be sold is

$$p(0) = -\frac{1}{4} \ln (2(0)^2 + 5) + 4.18 = 3.78$$

Now Work Problem 49 ⬤

EXAMPLE 28  An Integral Involving Two Forms

Find $\int \left( \frac{1}{(1 - w)^2} + \frac{1}{w - 1} \right) \, dw$.

Solution:

$$\int \left( \frac{1}{(1 - w)^2} + \frac{1}{w - 1} \right) \, dw = \int (1 - w)^{-2} \, dw + \int \frac{1}{w - 1} \, dw$$

$$= -1 \int (1 - w)^{-2} [ -dw ] + \int \frac{1}{w - 1} \, dw$$

The first integral has the form $\int u^{-2} \, du$, and the second has the form $\int \frac{1}{v} \, dv$. Thus,

$$\int \left( \frac{1}{(1 - w)^2} + \frac{1}{w - 1} \right) \, dw = \frac{(1 - w)^{-1}}{-1} + \ln |w - 1| + C$$

$$= \frac{1}{1 - w} + \ln |w - 1| + C$$
## Problems 13.4

In Problems 1–78, find the indefinite integrals.

1. \( \int (x + 5)^3 \, dx \)
2. \( \int 15(x + 2)^4 \, dx \)
3. \( \int 2x^2 + 3 \, dx \)
4. \( \int (4x + 3)(2x^2 + 3x + 1) \, dx \)
5. \( \int (3y^2 + 6y)(y^3 + 3y^2 + 1)^{2/3} \, dy \)
6. \( \int (15t^2 - 6t + 1)(5t^3 - 3t^2 + t^{1/3}) \, dt \)
7. \( \int \frac{5}{(3x - 1)^5} \, dx \)
8. \( \sqrt{7x + 3} \, dx \)
9. \( \int \frac{1}{\sqrt{x} - 5} \, dx \)
10. \( \int (7x - 6)^4 \, dx \)
11. \( \int 5x^3 + 7 \, dx \)
12. \( \int u(5u^2 - 9)^{14} \, du \)
13. \( \int x\sqrt{3 + 5x^2} \, dx \)
14. \( \int (4 - 5x)^9 \, dx \)
15. \( \int 4x^4(27 + x^6)^{1/3} \, dx \)
16. \( \int 3e^{3x} \, dx \)
17. \( \int 5e^{3t+7} \, dt \)
18. \( \int (3t + 1)e^{3t+2r+1} \, dt \)
19. \( \int -3w^2 e^{-w^3} \, dw \)
20. \( \int x^2 e^{x^2} \, dx \)
21. \( \int x^3 e^{4x} \, dx \)
22. \( \int 4e^{-3x} \, dx \)
23. \( \int 24x^5 e^{-2x^7} \, dx \)
24. \( \int \frac{1}{x + 5} \, dx \)
25. \( \int \frac{3x^2 + 4x^3}{x^3 + x^4} \, dx \)
26. \( \int \frac{6x^2 - 6x}{1 - 3x^2 + 2x^3} \, dx \)
27. \( \int \frac{8x}{(z^2 - 5)^7} \, dz \)
28. \( \int \frac{3}{(5v - 1)^4} \, dv \)
29. \( \int \frac{3}{1 + 2y} \, dy \)
30. \( \int \frac{3}{1 + 2y} \, dy \)
31. \( \int \frac{x^2}{s^3 + 5} \, ds \)
32. \( \int \frac{32x^3}{4x^4 + 9} \, dx \)
33. \( \int \frac{5}{4 - 2x} \, dx \)
34. \( \int \frac{7t}{5t^2 - 6} \, dt \)
35. \( \int \frac{1}{\sqrt{ax^3 + b}} \, dx \)
36. \( \int \frac{1}{(3x)^5} \, dx \)
37. \( \int \frac{9}{1 - 3x} \, dx \)
38. \( \int \frac{1}{\sqrt{x} + b} \, dx \)
39. \( \int 2y^3 e^{x^4+1} \, dy \)
40. \( \int 2\sqrt{2x - 1} \, dx \)
41. \( \int \frac{x^2 + x + 1}{\sqrt{x^3 + 2x^2 + 3x}} \, dx \)
42. \( \int (e^{-5x} + 2e^x) \, dx \)
43. \( \int 4\sqrt{y + 1} \, dy \)
44. \( \int 4\sqrt{y + 1} \, dy \)
45. \( \int (8x + 10)(7 - 2x^2 - 5x^3) \, dx \)
46. \( \int 2ye^{3y^2} \, dy \)
47. \( \int 6x^2 + 8 \frac{3}{x^2 + 4x} \, dx \)
48. \( \int (e^x + 2e^{-3x} - e^{5x}) \, dx \)
49. \( \int \frac{16s - 4}{3} + 4 \, ds \)
50. \( \int 6t^2 + 4 \, dt \)
51. \( \int x(2x^2 + 1)^{-1} \, dx \)
52. \( \int (45w^4 + 18w^2 + 12)(3w^5 + 2w^3 + 4)^{-3} \, dw \)
53. \( \int -(2 - 2x)(x^3 - x^2 + 1)^{-2} \, dx \)
54. \( \int \frac{3}{2}(v - 2)e^{2-vx+2v} \, dv \)
55. \( \int (2x^3 + x)(x^4 + x^2) \, dx \)
56. \( \int (e^{x^2})^2 \, dx \)
57. \( \int \frac{9 + 18x}{(5 - x - x^2)^2} \, dx \)
58. \( \int (e^x - e^{-x})^2 \, dx \)
59. \( \int x(2x + 1)e^{x^3+3x^2-4} \, dx \)
60. \( \int (a^3 - ae^{-3a^2}) \, du \)
61. \( \int x\sqrt{8 - 5x^2} \, dx \)
62. \( \int e^{ax} \, dx \)
63. \( \int \sqrt{2x - 1} \, dx \)
64. \( \int \frac{x^4}{e^{4x}} \, dx \)
65. \( \int (x^2 + 1)^2 \, dx \)
66. \( \int \left[ \frac{x(x^2 - 16)}{2x + 5} \right] \, dx \)
67. \( \int \left( \frac{x}{x^2 + 1} + \frac{x}{(x^2 + 1)^2} \right) \, dx \)
68. \( \int \left[ \frac{3}{x - 1} + \frac{1}{(x - 1)^2} \right] \, dx \)
69. \( \int \left[ \frac{2}{4x + 1} - (4x^2 - 8x^5)(x^3 - x^5)^{-8} \right] \, dx \)
70. \( \int (r^3 + 5)^2 \, dr \)
71. \( \int \left[ \frac{3x - 1}{x^3 + 3} \right] \, dx \)
72. \( \int \left[ \frac{x}{7x^2 + 2 - (x^3 + 2)^4} \right] \, dx \)
73. \( \int \frac{e^{x^2}}{\sqrt{x}} \, dx \)
74. \( \int (e^x - 3x) \, dx \)
75. \( \int \frac{1 + e^{2x}}{4e^{xt}} \, dx \)
76. \( \int \frac{2 + 1}{t^2 \sqrt{t^4 + 9} \, dt \}
77. \( \int \frac{4x + 3}{2x^2 + 3x} \ln(2x^2 + 3x) \, dx \)
78. \( \int \frac{1}{2\sqrt{x} + 3x} \, dx \)

In Problems 79–82, find \( y \) subject to the given conditions.

79. \( y' = (3 - 2x)^2; \quad y(0) = 1 \)
80. \( y' = \frac{x}{x^2 + 1}; \quad y(1) = 0 \)
81. \( y'' = \frac{1}{x^2}; \quad y'(2) = 3, y(1) = 2 \)
82. \( y'' = (x + 1)^{1/2}; \quad y'(8) = 19, y(24) = \frac{2572}{3} \)
83. **Real Estate** The rate of change of the value of a house in Djerba, Tunisia, that cost \$350,000 to build can be modeled by \( \frac{dv}{dt} = 8e^{0.05t} \), where \( t \) is the time in years since the house was built and \( V \) is the value (in thousands of dollars) of the house. Find \( V(t) \).
84. Oxygen in Capillary In a discussion of the diffusion of oxygen from capillaries, concentrated cylinders of radius \( r \) are used as a model for a capillary. The concentration \( C \) of oxygen in the capillary is given by

\[
C = \int \left( \frac{Rr}{2K} + \frac{B_1}{r} \right) \, dr
\]

where \( R \) is the constant rate at which oxygen diffuses from the capillary, and \( K \) and \( B_1 \) are constants. Find \( C \). (Write the constant of integration as \( B_2 \).)

85. Find \( f(2) \) if \( f \left( \frac{1}{3} \right) = 2 \) and \( f'(x) = e^{3x} + 2 - 3x \).

13.5 Techniques of Integration

We turn now to some more difficult integration problems.

When integrating fractions, sometimes a preliminary division is needed to get familiar integration forms, as the next example shows.

**EXAMPLE 29** Preliminary Division before Integration

a. Find \( \int \frac{x^3 + x}{x^2} \, dx \).

**Solution:** A familiar integration form is not apparent. However, we can break up the integrand into two fractions by dividing each term in the numerator by the denominator. We then have

\[
\int \frac{x^3 + x}{x^2} \, dx = \int \left( \frac{x^3}{x^2} + \frac{x}{x^2} \right) \, dx = \int \left( x + \frac{1}{x} \right) \, dx
\]

\[
= x^2 + \ln |x| + C
\]

b. Find \( \int \frac{2x^3 + 3x^2 + x + 1}{2x + 1} \, dx \).

**Solution:** Here the integrand is a quotient of polynomials in which the degree of the numerator is greater than or equal to that of the denominator. In such a situation we first use long division. Recall that if \( f \) and \( g \) are polynomials, with the degree of \( f \) greater than or equal to the degree of \( g \), then long division allows us to find (uniquely) polynomials \( q \) and \( r \), where either \( r \) is the zero polynomial or the degree of \( r \) is strictly less than the degree of \( g \), satisfying

\[
\frac{f}{g} = q + \frac{r}{g}
\]

Using an obvious, abbreviated notation, we see that

\[
\int \frac{f}{g} = \int \left( q + \frac{r}{g} \right) = \int q + \int \frac{r}{g}
\]

Since integrating a polynomial is easy, we see that integrating rational functions reduces to the task of integrating proper rational functions—those for which the degree of the numerator is strictly less than the degree of the denominator. In this case we obtain

\[
\int \frac{2x^3 + 3x^2 + x + 1}{2x + 1} \, dx = \int \left( x^2 + x + \frac{1}{2} \right) \, dx
\]

\[
= x^3 + \frac{x^2}{2} + \frac{1}{2} \ln |2x + 1| + C
\]

Now Work Problem 1 <

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Section 13.5  Techniques of Integration

EXAMPLE 30  Indefinite Integrals

a. Find \( \int \frac{1}{\sqrt{x}(\sqrt{x} - 2)^3} \, dx \).

Solution: We can write this integral as \( \int \frac{(\sqrt{x} - 2)^{-3}}{\sqrt{x}} \, dx \). Let us try the power rule for integration with \( u = \sqrt{x} - 2 \). Then \( du = \frac{1}{2\sqrt{x}} \, dx \), so that \( \frac{1}{\sqrt{x}} \, dx = 2 \, du \), and

\[
\int \frac{(\sqrt{x} - 2)^{-3}}{\sqrt{x}} \, dx = 2 \int u^{-3} \, du = 2 \left( \frac{u^{-2}}{-2} \right) + C
\]

\[
= -\frac{1}{u^2} + C = -\frac{1}{(\sqrt{x} - 2)^2} + C
\]

b. Find \( \int \frac{1}{x \ln x} \, dx \).

Solution: If \( u = \ln x \), then \( du = \frac{1}{x} \, dx \), and

\[
\int \frac{1}{x \ln x} \, dx = \int \frac{1}{\ln x} \left( \frac{1}{x} \, dx \right) = \int \frac{1}{u} \, du
\]

\[
= \ln |u| + C = \ln |\ln x| + C
\]

c. Find \( \int \frac{5}{w(\ln w)^{3/2}} \, dw \).

Solution: If \( u = \ln w \), then \( du = \frac{1}{w} \, dw \). Applying the power rule for integration, we have

\[
\int \frac{5}{w(\ln w)^{3/2}} \, dw = 5 \int (\ln w)^{-3/2} \left( \frac{1}{w} \, dw \right)
\]

\[
= 5 \int u^{-3/2} \, du = 5 \cdot \frac{u^{-1/2}}{-1/2} + C
\]

\[
= -10 \cdot \frac{1}{u^{1/2}} + C = -\frac{10}{(\ln w)^{1/2}} + C
\]

Now Work Problem 23 ⊳

Integrating \( b^u \)

In Section 13.4, we integrated an exponential function to the base \( e \):

\[
\int e^u \, du = e^u + C
\]

Now let us consider the integral of an exponential function with an arbitrary base, \( b \).

\[
\int b^u \, du
\]

To find this integral, we first convert to base \( e \) using

\[
b^u = e^{(\ln b)u}
\]

(\text{as we did in many differentiation examples too). Example 31 will illustrate.}
12. The rate of change of Syria’s total fertility rate (average number of children born to each woman) can be approximated by the function \( f'(t) = -0.113(0.971)^{t-2} \) where \( t = 0 \) corresponds to 2000. Find the total fertility rate function \( f(t) \) if the total fertility rate in 2011 was 2.94.

Source: Based on data from the CIA World Factbook.

**EXAMPLE 31** An Integral Involving \( b^u \)

Find \( \int 2^{3-x} \, dx \).

**Solution:**

**Strategy** We want to integrate an exponential function to the base 2. To do this, we will first convert from base 2 to base \( e \) by using Equation (1).

\[
\int 2^{3-x} \, dx = \int e^{(\ln 2)(3-x)} \, dx
\]

The integrand of the second integral is of the form \( e^u \), where \( u = (\ln 2)(3-x) \). Since \( du = -\ln 2 \, dx \), we can solve for \( dx \) and write

\[
\int e^{(\ln 2)(3-x)} \, dx = -\frac{1}{\ln 2} \int e^u \, du
\]

\[
= -\frac{1}{\ln 2} e^u + C = -\frac{1}{\ln 2} e^{(\ln 2)(3-x)} + C = -\frac{1}{\ln 2} 2^{3-x} + C
\]

Thus,

\[
\int 2^{3-x} \, dx = -\frac{1}{\ln 2} 2^{3-x} + C
\]

Notice that we expressed our answer in terms of an exponential function to the base 2, the base of the original integrand.

**Now Work Problem 27**

Generalizing the procedure described in Example 31, we can obtain a formula for integrating \( b^u \):

\[
\int b^u \, du = \int e^{(\ln b)u} \, du
\]

\[
= \frac{1}{\ln b} \int e^{(\ln b)u} d((\ln b)u)
\]

\[
= \frac{1}{\ln b} e^{(\ln b)u} + C
\]

\[
= \frac{1}{\ln b} b^u + C
\]

Hence, we have

\[
\int b^u \, du = \frac{1}{\ln b} b^u + C
\]

Applying this formula to the integral in Example 31 gives

\[
\int 2^{3-x} \, dx \quad \quad b = 2, u = 3 - x
\]

\[
= -\frac{1}{\ln 2} 2^{3-x} + C
\]

which is the same result that we obtained before.
Application of Integration

We will now consider an application of integration that relates a consumption function to the marginal propensity to consume.

**EXAMPLE 32 Finding a Consumption Function from Marginal Propensity to Consume**

Suppose that the marginal propensity to consume for Bahrain is given by

$$\frac{dC}{dI} = \frac{3}{4} - \frac{1}{2\sqrt{3I}}$$

where consumption $C$ is a function of national income $I$. Here $I$ is expressed in large denominations of money. Determine the consumption function for Bahrain if it is known that consumption is 10 ($C = 10$) when $I = 12$.

**Solution:** Since the marginal propensity to consume is the derivative of $C$, we have

$$C = C(I) = \int \left( \frac{3}{4} - \frac{1}{2\sqrt{3I}} \right) dI = \int \frac{3}{4} dI - \frac{1}{2} \int (3I)^{-1/2} dI$$

If we let $u = 3I$, then $du = 3 dI = d(3I)$, and

$$C = \frac{3}{4}I - \frac{1}{2} \frac{1}{3} \int (3I)^{-1/2} d(3I)$$

$$C = \frac{3}{4}I - \frac{1}{6} \frac{1}{\sqrt{3I}} + K$$

When $I = 12$, $C = 10$, so

$$10 = \frac{3}{4}12 - \frac{\sqrt{3(12)}}{3} + K$$

$$10 = 9 - 2 + K$$

Thus, $K = 3$, and the consumption function is

$$C = \frac{3}{4}I - \frac{\sqrt{3I}}{3} + 3$$

Now Work Problem 59 ⊳

**PROBLEMS 13.5**

*In Problems 1–53, determine the indefinite integrals.*

1. $\int \frac{2x^6 + 8x^3 - 4x}{2x^2} \, dx$
2. $\int \frac{9x^2 + 5}{3x} \, dx$
3. $\int \frac{x}{\sqrt{x^2 + 2}} \, dx$
4. $\int \frac{2x^3}{\sqrt{3x^2 + 4}} + 1 \, dx$
5. $\int \frac{3}{\sqrt{4 - 5x}} \, dx$
6. $\int \frac{2xe^x}{e^x - 2} \, dx$
7. $\int 4x^2 \, dx$
8. $\int 5' \, dt$
9. $\int 2x(x^2 - e^{x^2}) \, dx$
10. $\int \frac{e^x + 1}{e^x} \, dx$
11. $\int \frac{6x^2 - 11x + 5}{3x - 1} \, dx$
12. $\int \frac{(3x + 2)(x - 4)}{x - 3} \, dx$
13. $\int \frac{5e^{3x}}{7e^{4x} + 4} \, dx$
14. $\int 6(e^{4 - 3x})^2 \, dx$
15. $\int \frac{5e^{13x}}{x^2} \, dx$
16. $\int \frac{2x^3 - 6x^3 + x - 2}{x - 2} \, dx$
17. $\int \frac{5x^3}{x^2 + 9} \, dx$
18. $\int \frac{5 - 4x^2}{3 + 2x} \, dx$
19. $\int \frac{5e^t}{1 + 3e^t} \, ds$
20. \[ \int \frac{5(x^{1/3} + 2)^4}{\sqrt{x^2}} \, dx \]  
21. \[ \int \frac{1}{x^{1/3} + \sqrt[x]{2}} \, dx \]  
22. \[ \int \sqrt{t - 3 - t\sqrt{t}} \, dt \]  
23. \[ \int \frac{\ln x}{x} \, dx \]  
24. \[ \int \frac{e^{x^2}}{x^2 + 1} \, dx \]  
25. \[ \int \frac{9x^4 - 6x^4 - e^x^3}{7x^2} \, dx \]  
26. \[ \int \frac{4}{x \ln (2x^2)} \, dx \]  
27. \[ \int 3\ln x \, dx \]  
28. \[ \int x^2 \sqrt{e^{x^2} + 1} \, dx \]  
29. \[ \int \frac{ax + b}{cx + d} \, dx \] 
30. \[ \int \frac{8}{(x + 3)\ln (x + 3)} \, dx \]  
31. \[ \int (e^{x^2} + x^2 - 2x) \, dx \]  
32. \[ \int \frac{x^3 + x^2 - x - 3}{x^2 - 3} \, dx \]  
33. \[ \int 4x \ln \sqrt{1 + x^2} \, dx \]  
34. \[ \int \frac{12x^3 \sqrt{(x^2 + 1)^3}}{x^4 + 1} \, dx \]  
35. \[ \int 3(x^2 + 2)^{-1/2} \sqrt{x^2 + 2} \, dx \]  
36. \[ \int \frac{x - x^2}{x^2 + 2x^{-1}} \, dx \]  
37. \[ \int \frac{x}{x^2 + 2x^{-1}} \, dx \]  
38. \[ \int \frac{2x^4 - 8x^3 - 6x^2 + 4}{x^3} \, dx \]  
39. \[ \int \frac{e^t - e^{-x}}{e^t + e^{-x}} \, dx \]  
40. \[ \int \frac{x}{x + 1} \, dx \]  
41. \[ \int \frac{2x}{(x^2 + 1)\ln (x^2 + 1)} \, dx \]  

\[
\frac{dc}{dq} = 10 - \frac{100}{q + 10}
\]

where \( c \) is the total cost in dollars when \( q \) units are produced.

(a) Determine the marginal cost when 40 units are produced.

(b) If fixed costs are $10,000, find the total cost of producing 40 units.

(c) Use the results of parts (a) and (b) and differentials to approximate the total cost of producing 42 units.

61. **Cost Function** The marginal-cost function for a manufacturer’s product is given by

\[
\frac{dc}{dq} = 100q^2 - 3998q + 60
\]

where \( c \) is the total cost in dollars when \( q \) units are produced.

(a) Determine the marginal cost when 40 units are produced.

(b) If fixed costs are $10,000, find the total cost of producing 40 units.

(c) Use the results of parts (a) and (b) and differentials to approximate the total cost of producing 42 units.

62. **Cost Function** Suppose the marginal-cost function for a manufacturer’s product is given by

\[
\frac{dc}{dq} = \frac{9}{10} \sqrt{q} \sqrt{0.04q^{3/4} + 4}
\]

where \( c \) is the total cost in dollars when \( q \) units are produced.

(a) Determine the marginal cost when 25 units are produced.

(b) Find the total cost of producing 25 units.

(c) Use the results of parts (a) and (b) and differentials to approximate the total cost of producing 23 units.
Objective

To motivate, by means of the concept of area, the definite integral as a limit of a special sum; to evaluate simple definite integrals by using a limiting process.

13.6 The Definite Integral

Figure 13.2 shows the region $R$ bounded by the lines $y = f(x) = 2x, y = 0$ (the $x$-axis), and $x = 1$. The region is simply a right triangle. If $b$ and $h$ are the lengths of the base and the height, respectively, then, from geometry, the area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}(1)(2) = 1$ square unit. (Henceforth, we will treat areas as pure numbers and write square unit only if it seems necessary for emphasis.) We will now find this area by another method, which, as we will see later, applies to more complex regions. This method involves the summation of areas of rectangles.

Let us divide the interval $[0, 1]$ on the $x$-axis into four subintervals of equal length by means of the equally spaced points $x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{2}{4}, x_3 = \frac{3}{4},$ and $x_4 = \frac{4}{4} = 1.$ (See Figure 13.3.) Each subinterval has length $\Delta x = \frac{1}{4}.$ These subintervals determine four subregions of $R$: $R_1, R_2, R_3,$ and $R_4$, as indicated.

With each subregion, we can associate a circumscribed rectangle (Figure 13.4)—that is, a rectangle whose base is the corresponding subinterval and whose height is the maximum value of $f(x)$ on that subinterval. Since $f$ is an increasing function, the maximum value of $f(x)$ on each subinterval occurs when $x$ is the right-hand endpoint. Thus, the areas of the circumscribed rectangles associated with regions $R_1, R_2, R_3,$ and $R_4$ are $\frac{1}{2}f(\frac{1}{4}), \frac{1}{2}f(\frac{2}{4}), \frac{1}{2}f(\frac{3}{4}),$ and $\frac{1}{2}f(\frac{4}{4}),$ respectively. The area of each rectangle is an approximation to the area of its corresponding subregion. Hence, the sum of the areas of these rectangles, denoted by $\overline{S}_4$ (read “$S$ upper bar sub 4” or “the fourth upper sum”), approximates the area $A$ of the triangle. We have

$$\overline{S}_4 = \frac{1}{2}f(\frac{1}{4}) + \frac{1}{2}f(\frac{2}{4}) + \frac{1}{2}f(\frac{3}{4}) + \frac{1}{2}f(\frac{4}{4})$$

$$= \frac{1}{2}(2(\frac{1}{4}) + 2(\frac{2}{4}) + 2(\frac{3}{4}) + 2(\frac{4}{4})) = \frac{5}{4}$$
You can verify that \( \bar{S}_4 = \sum_{i=1}^{4} f(x_i) \Delta x \). The fact that \( \bar{S}_4 \) is greater than the actual area of the triangle might have been expected, since \( \bar{S}_4 \) includes areas of shaded regions that are not in the triangle. (See Figure 13.4.)

On the other hand, with each subregion we can also associate an inscribed rectangle (Figure 13.5)—that is, a rectangle whose base is the corresponding subinterval, but whose height is the minimum value of \( f(x) \) on that subinterval. Since \( f \) is an increasing function, the minimum value of \( f(x) \) on each subinterval will occur when \( x \) is the left-hand endpoint. Thus, the areas of the four inscribed rectangles associated with \( R_1, R_2, R_3, \) and \( R_4 \) are \( \frac{1}{2} f(0), \frac{1}{2} f(\frac{\Delta x}{2}), \frac{1}{2} f(\frac{\Delta x}{2}), \) and \( \frac{1}{2} f(\frac{\Delta x}{2}) \), respectively. Their sum, denoted \( S_4 \) (read “the first lower sum” or “the fourth lower sum”), is also an approximation to the area \( A \) of the triangle. We have

\[
S_4 = \frac{1}{4} f(0) + \frac{1}{4} f(\frac{\Delta x}{2}) + \frac{1}{4} f(\frac{\Delta x}{2}) + \frac{1}{4} f(\frac{\Delta x}{2}) = \frac{1}{4} (2(0) + 2 (\frac{\Delta x}{2}) + 2 (\frac{\Delta x}{2}) + 2 (\frac{\Delta x}{2})) = \frac{3}{4}
\]

Using summation notation, we can write \( S_4 = \sum_{i=0}^{3} f(x_i) \Delta x \). Note that \( S_4 \) is less than the area of the triangle, because the rectangles do not account for the portion of the triangle that is not shaded in Figure 13.5.

Since \( \frac{3}{4} = S_4 \leq A \leq \bar{S}_4 = \frac{5}{4} \)

we say that \( S_4 \) is an approximation to \( A \) from below and \( \bar{S}_4 \) is an approximation to \( A \) from above.

If \([0, 1]\) is divided into more subintervals, we expect that better approximations to \( A \) will occur. To test this, let us use six subintervals of equal length \( \Delta x = \frac{1}{6} \). Then \( S_6 \), the total area of six inscribed rectangles (see Figure 13.6), and \( \bar{S}_6 \), the total area of six circumscribed rectangles (see Figure 13.7), are

\[
S_6 = \frac{1}{6} f(0) + \frac{1}{6} f(\frac{\Delta x}{6}) + \frac{1}{6} f(\frac{2\Delta x}{6}) + \frac{1}{6} f(\frac{4\Delta x}{6}) + \frac{1}{6} f(\frac{5\Delta x}{6}) + \frac{1}{6} f(\frac{6\Delta x}{6}) = \frac{1}{6} (2(0) + 2 (\frac{\Delta x}{6}) + 2 (\frac{2\Delta x}{6}) + 2 (\frac{4\Delta x}{6}) + 2 (\frac{5\Delta x}{6}) + 2 (\frac{6\Delta x}{6})) = \frac{7}{6}
\]

and

\[
\bar{S}_6 = \frac{1}{6} f(0) + \frac{1}{6} f(\frac{\Delta x}{6}) + \frac{1}{6} f(\frac{2\Delta x}{6}) + \frac{1}{6} f(\frac{4\Delta x}{6}) + \frac{1}{6} f(\frac{5\Delta x}{6}) + \frac{1}{6} f(\frac{6\Delta x}{6}) = \frac{1}{6} (2(0) + 2 (\frac{\Delta x}{6}) + 2 (\frac{2\Delta x}{6}) + 2 (\frac{4\Delta x}{6}) + 2 (\frac{5\Delta x}{6}) + 2 (\frac{6\Delta x}{6})) = \frac{5}{6}
\]

Note that \( S_6 \leq A \leq \bar{S}_6 \), and, with appropriate labeling, both \( S_6 \) and \( \bar{S}_6 \) will be of the form \( \sum f(x) \Delta x \). Clearly, using six subintervals gives better approximations to the area than does four subintervals, as expected.

More generally, if we divide \([0, 1]\) into \( n \) subintervals of equal length \( \Delta x \), then \( \Delta x = 1/n \), and the endpoints of the subintervals are \( x = 0, 1/n, 2/n, \ldots, (n-1)/n, \) and \( n/\Delta x = 1 \). (See Figure 13.8.) The endpoints of the \( k \)th subinterval, for \( k = 1, \ldots, n \), are \( (k-1)/n \) and \( k/n \) and the maximum value of \( f \) occurs at the right-hand endpoint \( k/n \). It follows that the area of the \( k \)th circumscribed rectangle is \( 1/n \cdot f(k/n) = 1/n \cdot 2(k/n) = 2k/n^2 \), for \( k = 1, \ldots, n \). The total area of all \( n \) circumscribed rectangles is

\[
\bar{S}_n = \sum_{k=1}^{n} f(k/n) \Delta x = \sum_{k=1}^{n} \frac{2k}{n^2}
\]

by factoring \( \frac{2}{n^2} \) from each term

\[
= \frac{2}{n^2} \cdot \frac{n(n+1)}{2} \quad \text{from Section 4.1}
\]

\[
= \frac{n+1}{n}
\]

(We recall that \( \sum_{k=1}^{n} k = 1 + 2 + \cdots + n \) is the sum of the first \( n \) positive integers and the formula used above was derived in Section 4.1 in anticipation of its application here.)
For *inscribed* rectangles, we note that the minimum value of \( f \) occurs at the left-hand endpoint, \((k - 1)/n\), of \([ (k - 1)/n, k/n] \), so that the area of the \( k \)th inscribed rectangle is \( 1/n \cdot f((k - 1)/n) = 1/n \cdot 2((k - 1)/n) = 2(k - 1)/n^2 \), for \( k = 1, \ldots, n \). The total area determined of all \( n \) inscribed rectangles (see Figure 13.9) is

\[
S_n = \sum_{k=1}^{n} f((k - 1)/n) \Delta x = \sum_{k=1}^{n} \frac{2(k - 1)}{n^2}
\]  

(2)

From Equations (1) and (2), we again see that both \( \overline{S}_n \) and \( \underline{S}_n \) are sums of the form

\[
\sum f(x) \Delta x, \text{ namely, } \underline{S}_n = \sum f \left( \frac{k}{n} \right) \Delta x \text{ and } \overline{S}_n = \sum f \left( \frac{k - 1}{n} \right) \Delta x.
\]

From the nature of \( \underline{S}_n \) and \( \overline{S}_n \), it seems reasonable—and it is indeed true—that

\[
\underline{S}_n \leq A \leq \overline{S}_n
\]

As \( n \) becomes larger, \( \underline{S}_n \) and \( \overline{S}_n \) become better approximations to \( A \). In fact, let us take the limits of \( \underline{S}_n \) and \( \overline{S}_n \) as \( n \) approaches \( \infty \) through positive integral values:

\[
\lim_{n \to \infty} \underline{S}_n = \lim_{n \to \infty} \frac{n - 1}{n} = \lim_{n \to \infty} \left( 1 - \frac{1}{n} \right) = 1
\]

\[
\lim_{n \to \infty} \overline{S}_n = \lim_{n \to \infty} \frac{n + 1}{n} = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right) = 1
\]

Since \( \underline{S}_n \) and \( \overline{S}_n \) have the same limit, namely,

\[
\lim_{n \to \infty} \underline{S}_n = \lim_{n \to \infty} \overline{S}_n = 1
\]

and since

\[
\underline{S}_n \leq A \leq \overline{S}_n
\]

we will take this limit to be the area of the triangle. Thus \( A = 1 \), which agrees with our prior finding. It is important to understand that here we developed a definition of the notion of area that is applicable to many different regions.

We call the common limit of \( \overline{S}_n \) and \( \underline{S}_n \), namely, 1, the *definite integral* of \( f(x) = 2x \) on the interval from \( x = 0 \) to \( x = 1 \), and we denote this quantity by writing

\[
\int_{0}^{1} 2x \, dx = 1
\]

(4)

The reason for using the term *definite integral* and the symbolism in Equation (4) will become apparent in the next section. The numbers 0 and 1 appearing with the integral sign \( \int \) in Equation (4) are called the *limits of integration*; 0 is the *lower limit* and 1 is the *upper limit*.

In general, for a function \( f \) defined on the interval from \( x = a \) to \( x = b \), where \( a < b \), we can form the sums \( \overline{S}_n \) and \( \underline{S}_n \), which are obtained by considering the maximum and
minimum values, respectively, on each of \( n \) subintervals of equal length \( \Delta x \). We can now state the following:

The common limit of \( S_n \) and \( S_n' \) as \( n \to \infty \), if it exists, is called the \textbf{definite integral} of \( f \) over \([a, b]\) and is written
\[
\int_a^b f(x) \, dx
\]

The numbers \( a \) and \( b \) are called \textbf{limits of integration}; \( a \) is the \textbf{lower limit} and \( b \) is the \textbf{upper limit}. The symbol \( x \) is called the \textbf{variable of integration} and \( f(x) \) is the \textbf{integrand}.

In terms of a limiting process, we have
\[
\sum f(x) \Delta x \to \int_a^b f(x) \, dx
\]

Two points must be made about the definite integral. First, the definite integral is the limit of a sum of the form \( \sum f(x) \Delta x \). In fact, we can think of the integral sign as an elongated “S,” the first letter of “Summation.” Second, for an arbitrary function \( f \) defined on an interval, we may be able to calculate the sums \( S_n \) and \( S_n' \) and determine their common limit if it exists. However, some terms in the sums may be negative if \( f(x) \) is negative at points in the interval. These terms are not areas of rectangles (an area is never negative), so the common limit may not represent an area. Thus, the \textbf{definite integral is nothing more than a real number; it may or may not represent an area}.

As we saw in Equation (3), \( \lim_{n \to \infty} S_n \) is equal to \( \lim_{n \to \infty} S_n' \). For an arbitrary function, this is not always true. However, for the functions that we will consider, these limits will be equal, and the definite integral will always exist. To save time, we will just use the \textbf{right-hand endpoint} of each subinterval in computing a sum. For the functions in this section, this sum will be denoted \( S_n \).

**EXAMPLE 33**  Computing an Area by Using Right-Hand Endpoints

Find the area of the region in the first quadrant bounded by \( f(x) = 4 - x^2 \) and the lines \( x = 0 \) and \( y = 0 \).

**Solution:** A sketch of the region appears in Figure 13.10. The interval over which \( x \) varies in this region is seen to be \([0, 2]\), which we divide into \( n \) subintervals of equal length \( \Delta x \). Since the length of \([0, 2]\) is 2, we take \( \Delta x = 2/n \). The endpoints of the subintervals are \( x_k = 0, 2/n, 2(2/n), \ldots, (n - 1)(2/n), \) and \( n(2/n) = 2 \), which are shown in Figure 13.11. The diagram also shows the corresponding rectangles obtained by using the right-hand endpoint of each subinterval. The area of the \( k \)th rectangle, for \( k = 1, \ldots, n \), is the product of its width, \( 2/n \), and its height, \( f(k(2/n)) = 4 - (2k/n)^2 \), which is the function value at the right-hand endpoint of its base. Summing these areas, we get

\[
S_n = \sum_{k=1}^{n} \left( 4 - \frac{8k^2}{n^2} \right) \Delta x = \sum_{k=1}^{n} \left( 4 \frac{1}{n} - \left( \frac{2k}{n} \right)^2 \right) \frac{2}{n}
\]
\[
= \sum_{k=1}^{n} \left( \frac{8}{n} - \frac{8k^2}{n^3} \right) = \sum_{k=1}^{n} \frac{8}{n} - \sum_{k=1}^{n} \frac{8k^2}{n^3} = \frac{8}{n} \sum_{k=1}^{n} 1 - \frac{8}{n} \sum_{k=1}^{n} k^2
\]
\[
= \frac{8}{n} - \frac{8n(n+1)(2n+1)}{6} = 8 - \frac{4}{3} \frac{(n+1)(2n+1)}{n^2}
\]

\footnote{Here we assume that the maximum and minimum values exist.}
The second line of the preceding computations uses basic summation manipulations as discussed in Section 4.1. The third line uses two specific summation formulas, also from Section 4.1: The sum of \( n \) copies of 1 is \( n \) and the sum of the first \( n \) squares is \( n(n+1)(2n+1)/6 \).

Finally, we take the limit of the \( S_n \) as \( n \to \infty \):

\[
\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left( 8 - \frac{4}{3} \frac{n+1}{n^2} \frac{(2n+1)}{2} \right) = 8 - \frac{4}{3} \lim_{n \to \infty} \left( \frac{2n^2 + 3n + 1}{n^2} \right) = 8 - \frac{4}{3} \lim_{n \to \infty} \left( \frac{2 \frac{3}{n} + \frac{1}{n^2}}{2} \right) = 8 - \frac{8}{3} = \frac{16}{3}
\]

Hence, the area of the region is \( \frac{16}{3} \).

Now Work Problem 7

**EXAMPLE 34** Evaluating a Definite Integral

Evaluate \( \int_0^2 (4 - x^2) \, dx \).

**Solution:** We want to find the definite integral of \( f(x) = 4 - x^2 \) over the interval \([0, 2]\).

Thus, we must compute \( \lim_{n \to \infty} S_n \). But this limit is precisely the limit \( \frac{16}{3} \) found in Example 33, so we conclude that

\[
\int_0^2 (4 - x^2) \, dx = \frac{16}{3}
\]

Now Work Problem 19

**EXAMPLE 35** Integrating a Function over an Interval

Integrate \( f(x) = x - 5 \) from \( x = 0 \) to \( x = 3 \); that is, evaluate \( \int_0^3 (x - 5) \, dx \).

**Solution:** We first divide \([0, 3]\) into \( n \) subintervals of equal length \( \Delta x = 3/n \). The endpoints are \( 0, 3/n, 2(3/n), \ldots, (n-1)(3/n), n(3/n) = 3 \). (See Figure 13.12.) Using right-hand endpoints, we form the sum and simplify

\[
S_n = \sum_{k=1}^{n} f \left( \frac{3k}{n} \right) \frac{3}{n} = \frac{3}{n} \sum_{k=1}^{n} \left( \frac{3k}{n} - 5 \right) = \frac{9}{n^2} \sum_{k=1}^{n} k - 15 \frac{n}{n} \sum_{k=1}^{n} 1 = \frac{9}{n^2} \left( \frac{n(n+1)}{2} \right) - 15 \frac{n}{n} = \frac{9}{2} \frac{n^2 + 1}{n} - 15 = \frac{9}{2} \left( 1 + \frac{1}{n} \right) - 15
\]

Taking the limit, we obtain

\[
\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left( \frac{9}{2} \left( 1 + \frac{1}{n} \right) - 15 \right) = \frac{9}{2} - 15 = -\frac{21}{2}
\]
Thus,
\[ \int_0^3 (x - 5) \, dx = -\frac{21}{2} \]

Note that the definite integral here is a negative number. The reason is clear from the graph of \( f(x) = x - 5 \) over the interval \([0, 3]\). (See Figure 13.13.) Since the value of \( f(x) \) is negative at each right-hand endpoint, each term in \( S_n \) must also be negative. Hence, \( \lim_{n \to \infty} S_n \), which is the definite integral, is negative.

Geometrically, each term in \( S_n \) is the negative of the area of a rectangle. (Refer again to Figure 13.13.) Although the definite integral is simply a number, here we can interpret it as representing the negative of the area of the region bounded by \( f(x) = x - 5, x = 0, x = 3, \) and the \( x \)-axis (\( y = 0 \)).

**Now Work Problem 17**

In Example 35, it was shown that the definite integral does not have to represent an area. In fact, there the definite integral was negative. However, if \( f \) is continuous and \( f(x) \geq 0 \) on \([a, b]\), then \( S_n \geq 0 \) for all values of \( n \). Therefore, \( \lim_{n \to \infty} S_n \geq 0 \), so
\[ \int_a^b f(x) \, dx \geq 0 \]
Furthermore, this definite integral gives the area of the region bounded by \( y = f(x), \ y = 0, \ x = a, \) and \( x = b \). (See Figure 13.14.)

Although the approach that we took to discuss the definite integral is sufficient for our purposes, it is by no means rigorous. The important thing to remember about the definite integral is that it is the limit of a special sum.

**TECHNOLOGY**

Here is a program for the TI-83 Plus graphing calculator that will estimate the limit of \( S_n \) as \( n \to \infty \) for a function \( f \) defined on \([a, b]\).

**PROGRAM:** RIGHTSUM

```
Lbl 1
Input “SUBINTV”,N
(B − A)/N → H
∅ → S
A + H → X
1 → I
Lbl 2
Y₁ + S → S
X + H → X
I + 1 → I
If I ≤ N
Goto 2
H*S → S
Disp S
Pause
Goto 1
```

**FIGURE 13.15** Values of \( S_n \) for \( f(x) = x - 5 \) on \([0, 3]\).

value of \( S_n \). Each time ENTER is pressed, the program repeats. In this way, a display of values of \( S_n \) for various numbers of subintervals may be obtained. Figure 13.15 shows values of \( S_n(n = 100, 1000, \) and \( 2000) \) for the function \( f(x) = x - 5 \) on the interval \([0, 3]\). As \( n \to \infty \), it appears that \( S_n \to -10.5 \). Thus, we estimate that
\[ \lim_{n \to \infty} S_n \approx -10.5 \]

Equivalently,
\[ \int_0^3 (x - 5) \, dx \approx -10.5 \]

which agrees with our result in Example 35.

It is interesting to note that the time required for an older calculator to compute \( S_{2000} \) in Figure 13.15 was in excess of 1.5 minutes. The time required on a TI-84 Plus is less than 1 minute.
13.7 The Fundamental Theorem of Integral Calculus

The Fundamental Theorem

Thus far, the limiting processes of both the derivative and definite integral have been considered separately. We will now bring these fundamental ideas together and develop the important relationship that exists between them. As a result, we will be able to evaluate definite integrals more efficiently.

The graph of a function \( f \) is given in Figure 13.16. Assume that \( f \) is continuous on the interval \([a, b]\) and that its graph does not fall below the \( x \)-axis. That is, \( f(x) \geq 0 \). From the preceding section, the area of the region below the graph and above the \( x \)-axis from \( x = a \) to \( x = b \) is given by \( \int_a^b f(x) \, dx \). We will now consider another way to determine this area.

Suppose that there is a function \( A = A(x) \), which we will refer to as an area function, that gives the area of the region below the graph of \( f \) and above the \( x \)-axis from \( a \) to \( x \) where \( a \leq x \leq b \). This region is shaded in Figure 13.17. Do not confuse \( A(x) \), which is an area, with \( f(x) \), which is the height of the graph at \( x \).
From its definition, we can state two properties of $A$ immediately:

1. $A(a) = 0$, since there is “no area” from $a$ to $a$
2. $A(b)$ is the area from $a$ to $b$; that is,

   $$A(b) = \int_a^b f(x) \, dx$$

If $x$ is increased by $h$ units, then $A(x + h)$ is the area of the shaded region in Figure 13.18. Hence, $A(x + h) - A(x)$ is the difference of the areas in Figures 13.18 and 13.17, namely, the area of the shaded region in Figure 13.19. For $h$ sufficiently close to zero, the area of this region is the same as the area of a rectangle (Figure 13.20) whose base is $h$ and whose height is some value $\bar{y}$ between $f(x)$ and $f(x + h)$. Here $\bar{y}$ is a function of $h$. Thus, on the one hand, the area of the rectangle is $A(x + h) - A(x)$, and, on the other hand, it is $h\bar{y}$, so

$$A(x + h) - A(x) = h\bar{y}$$

Equivalently,

$$\frac{A(x + h) - A(x)}{h} = \bar{y} \quad \text{dividing by } h$$

Since $\bar{y}$ is between $f(x)$ and $f(x + h)$, it follows that as $h \to 0$, $\bar{y}$ approaches $f(x)$, so

$$\lim_{h \to 0} \frac{A(x + h) - A(x)}{h} = f(x) \quad (1)$$

But the left side is merely the derivative of $A$. Thus, Equation (1) becomes

$$A'(x) = f(x)$$

We conclude that the area function $A$ has the additional property that its derivative $A'$ is $f$. That is, $A$ is an antiderivative of $f$. Now, suppose that $F$ is any antiderivative of $f$. Then, since both $A$ and $F$ are antiderivatives of the same function, they differ at most by a constant $C$:

$$A(x) = F(x) + C \quad (2)$$

Recall that $A(a) = 0$. So, evaluating both sides of Equation (2) when $x = a$ gives

$$0 = F(a) + C$$

so that

$$C = -F(a)$$

Thus, Equation (2) becomes

$$A(x) = F(x) - F(a) \quad (3)$$

If $x = b$, then, from Equation (3),

$$A(b) = F(b) - F(a) \quad (4)$$

But recall that

$$A(b) = \int_a^b f(x) \, dx \quad (5)$$

From Equations (4) and (5), we get

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

A relationship between a definite integral and antidifferentiation has now become clear. To find $\int_a^b f(x) \, dx$, it suffices to find an antiderivative of $f$, say, $F$, and subtract the value of $F$ at the lower limit $a$ from its value at the upper limit $b$. We assumed here that $f$ was continuous and $f(x) \geq 0$ so that we could appeal to the concept of an area. However, our result is true for any continuous function\footnote{If $f$ is continuous on $[a, b]$, it can be shown that $\int_a^b f(x) \, dx$ does indeed exist.} and is known as the \textbf{Fundamental Theorem of Integral Calculus}.  \[end]
The definite integral is a number, and an indefinite integral is a function.

**Fundamental Theorem of Integral Calculus**

If \( f \) is continuous on the interval \([a, b]\) and \( F \) is any antiderivative of \( f \) on \([a, b]\), then

\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]

It is important that you understand the difference between a definite integral and an indefinite integral. The definite integral \( \int_a^b f(x) \, dx \) is a number defined to be the limit of a sum. The Fundamental Theorem states that the indefinite integral \( \int f(x) \, dx \) (the most general antiderivative of \( f \)), which is a function of \( x \) related to the differentiation process, can be used to determine this limit.

Suppose we apply the Fundamental Theorem to evaluate \( \int_0^2 (4 - x^2) \, dx \). Here \( f(x) = 4 - x^2 \), \( a = 0 \), and \( b = 2 \). Since an antiderivative of \( 4 - x^2 \) is \( F(x) = 4x - (x^3/3) \), it follows that

\[
\int_0^2 (4 - x^2) \, dx = F(2) - F(0) = \left( 8 - \frac{8}{3} \right) - 0 = \frac{16}{3}
\]

This confirms our result in Example 34 of Section 13.6. If we had chosen \( F(x) \) to be \( 4x - (x^3/3) + C \), then we would have

\[
F(2) - F(0) = \left[ \left( 8 - \frac{8}{3} \right) + C \right] - [0 + C] = \frac{16}{3}
\]

as before. Since the choice of the value of \( C \) is immaterial, for convenience we will always choose it to be 0, as originally done. Usually, \( F(b) - F(a) \) is abbreviated by writing

\[
F(b) - F(a) = F(x)|_a^b
\]

Since \( F \) in the Fundamental Theorem of Calculus is any antiderivative of \( f \) and \( \int f(x) \, dx \) is the most general antiderivative of \( f \), it showcases the notation to write

\[
\int_a^b f(x) \, dx = \left( \int f(x) \, dx \right)|_a^b
\]

Using the \( |_a^b \) notation, we have

\[
\int_0^2 (4 - x^2) \, dx = \left( 4x - \frac{x^3}{3} \right)|_0^2 = \left( 8 - \frac{8}{3} \right) - 0 = \frac{16}{3}
\]

**EXAMPLE 36** Applying the Fundamental Theorem

The rate of production of electricity in Morocco for the years 2000 to 2006, measured in billions of kilowatt hours per year, can be modeled by \( E(t) = 1201.4e^{0.0875t} \) where \( t \) is in years. What is the total amount of electricity produced from 2003 to 2006?

**Solution:** The total amount of electricity produced is given by summing (integrating) the (instantaneous) production rate between year 3 and year 6:

\[
\int_3^6 1.201e^{0.0875t} \, dt = 1201.4(0.0875)e^{0.0875t}
\]

\[
= 10.512(e^{0.0875(6)} - e^{0.0875(3)})|_3^6
\]

\[
\approx 5358.69
\]

**APPLY IT**

14. The income (in dollars) from a fast-food chain in Beirut is increasing at a rate of \( f(t) = 10,000e^{0.02t} \), where \( t \) is in years. Find \( \int_3^6 10,000e^{0.02t} \, dt \), which gives the total income for the chain between the third and sixth years.
Therefore, approximately 5358.69 billion kilowatt hours of electricity were produced from 2003 and 2006 in Morocco.

Now Work Problem 1 ◁

Properties of the Definite Integral

For $\int_a^b f(x) \, dx$, we have assumed that $a < b$. We now define the cases in which $a > b$ or $a = b$. First,

If $a > b$, then $\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$.

That is, interchanging the limits of integration changes the integral’s sign. For example,

$$\int_2^0 (4 - x^2) \, dx = -\int_0^2 (4 - x^2) \, dx$$

If the limits of integration are equal, we have

$$\int_a^a f(x) \, dx = 0$$

Some properties of the definite integral deserve mention. The first of the properties that follow restates more formally our comment from the preceding section concerning area.

Properties of the Definite Integral

1. If $f$ is continuous and $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x) \, dx$ can be interpreted as the area of the region bounded by the curve $y = f(x)$, the $x$-axis, and the lines $x = a$ and $x = b$.

2. $\int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx$, where $k$ is a constant

3. $\int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$

Properties 2 and 3 are similar to rules for indefinite integrals because a definite integral may be evaluated by the Fundamental Theorem in terms of an antiderivative. Two more properties of definite integrals are as follows.

4. $\int_a^b f(x) \, dx = \int_a^b f(t) \, dt$

The variable of integration is a “dummy variable” in the sense that any other variable produces the same result—that is, the same number.

To illustrate property 4, you can verify, for example, that

$$\int_0^2 x^2 \, dx = \int_0^2 t^2 \, dt$$

5. If $f$ is continuous on an interval $I$ and $a$, $b$, and $c$ are in $I$, then

$$\int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx$$
Property 5 means that the definite integral over an interval can be expressed in terms of definite integrals over subintervals. Thus,

\[
\int_0^2 (4 - x^2) \, dx = \int_0^1 (4 - x^2) \, dx + \int_1^2 (4 - x^2) \, dx
\]

We will look at some examples of definite integration now and compute some areas in Section 13.8.

**EXAMPLE 37** Using the Fundamental Theorem

Find \( \int_0^1 \frac{x^3}{\sqrt{1 + x^4}} \, dx \).

**Solution:** To find an antiderivative of the integrand, we will apply the power rule for integration:

\[
\int_0^1 \frac{x^3}{\sqrt{1 + x^4}} \, dx = \int_0^1 x^3 (1 + x^4)^{-1/2} \, dx
\]

\[
= \frac{1}{4} \int_0^1 (1 + x^4)^{-1/2} \, d(1 + x^4) = \left[ \frac{1}{4} \frac{(1 + x^4)^{1/2}}{2} \right]_0^1
\]

\[
= \frac{1}{2} \left( 1 + x^4 \right)^{1/2} \bigg|_0^1 = \frac{1}{2} \left( (2)^{1/2} - (1)^{1/2} \right)
\]

\[
= \frac{1}{2} (\sqrt{2} - 1)
\]

**CAUTION**

In Example 37, the value of the antiderivative \( \frac{1}{2}(1 + x^4)^{1/2} \) at the lower limit 0 is \( \frac{1}{2}(1)^{1/2} \). Do not assume that an evaluation at the limit zero will yield 0.

**EXAMPLE 38** Evaluating Definite Integrals

a. Find \( \int_1^2 [4t^{1/3} + t(t^2 + 1)^3] \, dt \).

**Solution:**

\[
\int_1^2 [4t^{1/3} + t(t^2 + 1)^3] \, dt = 4 \int_1^2 t^{1/3} \, dt + \frac{1}{2} \int_1^2 (t^2 + 1)^3 \, dt(t^2 + 1)
\]

\[
= (4) \frac{t^{4/3}}{4} \bigg|_1^2 + \left( \frac{1}{2} \right) \frac{(t^2 + 1)^4}{4} \bigg|_1^2
\]

\[
= 3(2^{4/3} - 1) + \frac{1}{8}(5^4 - 2^4)
\]

\[
= 3 \cdot 2^{4/3} - 3 + \frac{609}{8}
\]

\[
= 6\sqrt{2} + \frac{585}{8}
\]

b. Find \( \int_0^1 e^{3t} \, dt \).

**Solution:**

\[
\int_0^1 e^{3t} \, dt = \frac{1}{3} \int_0^1 e^{3t} \, d(3t)
\]

\[
= \left( \frac{1}{3} \right) e^{3t} \bigg|_0^1 = \frac{1}{3}(e^3 - e^0) = \frac{1}{3}(e^3 - 1)
\]

**Now Work Problem 13 ⊳**

**Now Work Problem 15 ⊳**
EXAMPLE 39 Finding and Interpreting a Definite Integral

Evaluate \( \int_{-2}^{1} x^3 \, dx \).

Solution:

\[
\int_{-2}^{1} x^3 \, dx = \left. \frac{x^4}{4} \right|_{-2}^{1} = \frac{1^4}{4} - \frac{(-2)^4}{4} = 1 - \frac{16}{4} = -\frac{15}{4}
\]

The reason the result is negative is clear from the graph of \( y = x^3 \) on the interval \([-2, 1]\). (See Figure 13.21.) For \(-2 \leq x < 0\), \( f(x) \) is negative. Since a definite integral is a limit of a sum of the form \( \Sigma f(x) \Delta x \), it follows that \( \int_{-2}^{0} x^3 \, dx \) is not only a negative number, but also the negative of the area of the shaded region in the third quadrant. On the other hand, \( \int_{0}^{1} x^3 \, dx \) is the area of the shaded region in the first quadrant, since \( f(x) \geq 0 \) on \([0, 1]\). The definite integral over the entire interval \([-2, 1]\) is the algebraic sum of these numbers, because, from property 5,

\[
\int_{-2}^{1} x^3 \, dx = \int_{-2}^{0} x^3 \, dx + \int_{0}^{1} x^3 \, dx
\]

Thus, \( \int_{-2}^{1} x^3 \, dx \) does not represent the area between the curve and the \( x \)-axis. However, if area is desired, it can be given by

\[
\left| \int_{-2}^{0} x^3 \, dx \right| + \int_{0}^{1} x^3 \, dx
\]

Now Work Problem 25 ☞

The Definite Integral of a Derivative

Since a function \( f \) is an antiderivative of \( f' \), by the Fundamental Theorem we have

\[
\int_{a}^{b} f'(x) \, dx = f(b) - f(a) \tag{6}
\]

But \( f'(x) \) is the rate of change of \( f \) with respect to \( x \). Hence, if we know the rate of change of \( f \) and want to find the difference in function values \( f(b) - f(a) \), it suffices to evaluate \( \int_{a}^{b} f'(x) \, dx \).

EXAMPLE 40 Total Sales

During an advertising campaign, Al Atal Nuts found that the rate of change of sales is given by

\[
S'(t) = 8.43 + 30e^{0.032t}
\]

where \( S \) is the amount of sales in thousands of dollars and \( t \) is the number of weeks that the campaign has been running. What are the total sales at the end of the sixth week of the advertising campaign?

Solution: The total sales amount at the end of 6 weeks is given by

\[
\int_{0}^{6} S'(t) \, dt = \int_{0}^{6} (8.43 + 30e^{0.032t}) \, dt
\]

\[
= \int_{0}^{6} 8.43 \, dt + \int_{0}^{6} 30e^{0.032t} \, dt
\]

\[
= 8.43 [t]_{0}^{6} + 30 \left( \frac{e^{0.032t}}{0.032} \right)_{0}^{6}
\]

\[
= 8.43 \cdot 6 + \frac{30}{0.032} \left( e^{0.032 \cdot 6} - e^{0.032 \cdot 0} \right)
\]

\[
= 50.58 + \frac{30}{0.032} \cdot (e^{0.192} - 1)
\]

\[
= 50.58 + \frac{30}{0.032} \cdot (1.218 - 1)
\]

\[
= 50.58 + 468.75 \cdot 0.218
\]

\[
= 50.58 + 101.625
\]

\[
= 152.205
\]
Section 13.7  The Fundamental Theorem of Integral Calculus

\[
= 8.43r + 937.5e^{0.032t}\bigg|_0^6 \\
= (8.43(6) + 937.5e^{0.032(6)}) - (8.43(0) + 937.5e^{0.032(0)}) \\
\approx 210.106
\]

That is, about $210,106.

\[\triangleq\]

**EXAMPLE 41  Finding a Change in Function Values by Definite Integration**

A manufacturer’s marginal-cost function is

\[
\frac{dc}{dq} = 0.6q + 2
\]

If production is presently set at \( q = 80 \) units per week, how much more would it cost to increase production to 100 units per week?

**Solution:** The total-cost function is \( c = c(q) \), and we want to find the difference \( c(100) - c(80) \). The rate of change of \( c \) is \( \frac{dc}{dq} \), so, by Equation (6),

\[
c(100) - c(80) = \int_{80}^{100} \frac{dc}{dq} \, dq = \int_{80}^{100} (0.6q + 2) \, dq
\]

\[
= \left[ \frac{0.6q^2}{2} + 2q \right]_{80}^{100} = [0.3q^2 + 2q]_{80}^{100}
\]

\[
= [0.3(100)^2 + 2(100)] - [0.3(80)^2 + 2(80)]
\]

\[
= 3200 - 2080 = 1120
\]

If \( c \) is in dollars, then the cost of increasing production from 80 units to 100 units is $1120.

Now Work Problem 55 \[\triangleq\]

**TECHNOLOGY**

Many graphing calculators have the capability to estimate the value of a definite integral. On a TI-83 Plus, to estimate

\[
\int_{\text{lower}}^{\text{upper}} (0.6q + 2) \, dq
\]

we use the “fnInt” command, as indicated in Figure 13.22. The four parameters that must be entered with this command are

<table>
<thead>
<tr>
<th>function to be integrated</th>
<th>variable of integration</th>
<th>lower limit</th>
<th>upper limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X^3 )</td>
<td>( X )</td>
<td>(-2)</td>
<td>( 1)</td>
</tr>
</tbody>
</table>

We see that the value of the definite integral is approximately 1120, which agrees with the result in Example 41. Similarly, to estimate

\[
\int_{-2}^{1} x^3 \, dx
\]

we enter

\[
\text{fnInt}(X^3, X, -2, 1)
\]

or, alternatively, if we first store \( x^3 \) as \( Y_1 \), we can enter

\[
\text{fnInt}(Y_1, X, -2, 1)
\]

In each case we obtain \(-3.75\), which agrees with the result in Example 39.
PROBLEMS 13.7

In Problems 1–41, evaluate the definite integral.

1. \( \int_0^3 5 \, dx \)
2. \( \int_1^5 (e + 3e^3) \, dx \)
3. \( \int_1^2 5x \, dx \)
4. \( \int_1^2 -5x \, dx \)
5. \( \int_1^3 (2x - 3) \, dx \)
6. \( \int_1^4 (4 - 9y) \, dy \)
7. \( \int_1^4 (y^2 + 4y + 4) \, dy \)
8. \( \int_1^4 (2t - 3t^2) \, dt \)
9. \( \int_2^{-1} (3w^2 - w - 1) \, dw \)
10. \( \int_8^9 dt \)
11. \( \int_1^3 3t^3 \, dt \)
12. \( \int_1^2 3x^2 \, dx \)
13. \( \int_1^2 \sqrt{x^3} \, dx \)
14. \( \int_1^{3/2} x^2 + x + 1 \, dx \)
15. \( \int_1^3 \frac{1}{x^2} \, dx \)
16. \( \int_2^{-2} (z + 1)^4 \, dz \)
17. \( \int_1^8 (x^{1/3} - x^{-1/3}) \, dx \)
18. \( \int_0^1 2x^2(x^3 - 1)^3 \, dx \)
19. \( \int_1^3 (x + 2)^3 \, dx \)
20. \( \int_1^8 4 \, dy \)
21. \( \int_2^3 \frac{2}{x} \, dx \)
22. \( \int_0^1 e^x \, dx \)
23. \( \int_2^e \frac{1}{x - 1} \, dx \)
24. \( \int_1^5 (3x^2 + 4x)(x^3 + 2x^2)^3 \, dx \)
25. \( \int_0^{20/3} 5x^2 e^x \, dx \)
26. \( \int_3^4 \frac{3}{(x + 3)^2} \, dx \)
27. \( \int_{-1/3}^{20/3} \sqrt{3x + 5} \, dx \)
28. \( \int_{1/3}^2 \sqrt{10 - 3p} \, dp \)
29. \( \int_1^{-1} q \sqrt{q^2 + 3} \, dq \)
30. \( \int_0^1 x^2 \sqrt{7x^3 + 1} \, dx \)
31. \( \int_0^b \sqrt{2x - \frac{x}{(x^2 + 1)^{2/3}}} \, dx \)
32. \( \int_0^1 \frac{2x^3 + x}{x^2 + x + 1} \, dx \)
33. \( \int_0^a (m + ny) \, dy \)
34. \( \int_1^8 |x| \, dx \)
35. \( \int_1^x \sqrt{3(x^2 + x^3 - x^{-6})} \, dx \)
36. \( \int_1^{20/3} 6x^2 - \frac{1}{\sqrt{2x}} \, dx \)
37. \( \int_1^4 (x + 1)e^{x^2 + 2x} \, dx \)
38. \( \int_1^{95} \frac{x}{\ln e^x} \, dx \)
39. \( \int_0^2 \frac{x^6 + 6x^3 + x^3 + 8x^2 + x + 5}{x^3 + 5x + 1} \, dx \)
40. \( \int_1^5 \frac{1}{1 + e^x} \, dx \) (Hint: Multiply the integrand by \( \frac{1}{1 + e^x} \))
41. \( \int_0^2 f(x) \, dx \), where \( f(x) = \begin{cases} 4x^2 & \text{if } 0 \leq x < \frac{1}{2} \\ 2x & \text{if } \frac{1}{2} \leq x \leq 2 \end{cases} \)
42. Evaluate \( \left( \int_1^3 x \, dx \right)^3 - \int_1^3 x^3 \, dx \).
43. Suppose \( f(x) = \int_1^8 \frac{1}{\sqrt{t}} \, dt \). Evaluate \( \int_1^8 f(x) \, dx \).
44. Evaluate \( \int_0^a e^x \, dx + \int_{-1}^{\sqrt{2}} \frac{1}{3} \, dx \).
45. If \( \int_1^2 f(x) \, dx = 5 \) and \( \int_1^2 f(x) \, dx = 2 \), find \( \int_1^2 f(x) \, dx \).
46. If \( \int_1^4 f(x) \, dx = 6 \), \( \int_1^4 f(x) \, dx = 5 \), and \( \int_1^3 f(x) \, dx = 2 \), find \( \int_2^3 f(x) \, dx \).
47. Suppose that \( f(x) = \int_1^x e^x - e^{x-1} \, dx \) where \( x > e \). Find \( f'(x) \).

48. **Severity Index**

In discussing traffic safety, Shonle considers how much acceleration a person can tolerate in a crash so that there is no major injury. The severity index is defined as

\[
\text{S.I.} = \int_0^T a^{5/2} \, dt
\]

where \( a \) (a Greek letter read “alpha”) is considered a constant involved with a weighted average acceleration, and \( T \) is the duration of the crash. Find the severity index.

49. **Statistics**

In statistics, the mean \( \mu \) (a Greek letter read “mu”) of the continuous probability density function \( f \) defined on the interval \([a, b]\) is given by

\[
\mu = \int_a^b xf(x) \, dx
\]

and the variance \( \sigma^2 \) (\( \sigma \) is a Greek letter read “sigma”) is given by

\[
\sigma^2 = \int_a^b (x - \mu)^2 f(x) \, dx
\]

Compute \( \mu \) and then \( \sigma^2 \) if \( a = 0, b = 1, \) and \( f(x) = 6(x - x^2) \).

50. **Distribution of Incomes**

The economist Pareto has stated an empirical law of distribution of higher incomes that gives the
The Fundamental Theorem of Integral Calculus

is a function such that

where A and B are constants, set up a definite integral that gives the total number of persons with incomes between a and b, where \( a < b \).

51. Biology In a discussion of gene mutation,\(^{10}\) the following integral occurs:

\[
\int_{0}^{10^{-4}} x^{-1/2} \, dx
\]

Evaluate this integral.

52. Biology In biology, problems frequently arise involving the transfer of a substance between compartments. An example is a transfer from the bloodstream to tissue. Evaluate the following integral, which occurs in a two-compartment diffusion problem:\(^{11}\)

\[
\int_{0}^{t} (e^{-a\tau} - e^{-b\tau}) \, d\tau
\]

Here, \( \tau \) (read “tau”) is a Greek letter; \( a \) and \( b \) are constants.

53. Demography For a certain small Arab population, suppose \( l \) is a function such that \( l(x) \) is the number of persons who reach the age of \( x \) in any year of time. This function is called a life table function. Under appropriate conditions, the integral

\[
\int_{a}^{b} l(t) \, dt
\]

gives the expected number of people in the population between the exact ages of \( a \) and \( b \), inclusive. If

\[
l(x) = 1000\sqrt{110 - x}
\]

for \( 0 \leq x \leq 110 \)
determine the number of people between the exact ages of 10 and 29, inclusive. Give your answer to the nearest integer, since fractional answers make no sense. What is the size of the population?

54. Mineral Consumption If \( C \) is the yearly consumption of a mineral at time \( t = 0 \), then, under continuous consumption, the total amount of the mineral used in the interval \([0, t]\) is

\[
\int_{0}^{C} e^{	au} \, d\tau
\]

where \( k \) is the rate of consumption. For a rare-earth mineral, it has been determined that \( C = 3000 \) units and \( k = 0.05 \). Evaluate the integral for these data.

55. Marginal Cost A manufacturer’s marginal-cost function is

\[
dc \over dq = 0.2q + 8
\]

If \( c \) is in dollars, determine the cost involved to increase production from 65 to 75 units.

56. Marginal Cost Repeat Problem 55 if

\[
dc \over dq = 0.004q^{2} - 0.5q + 50
\]

and production increases from 90 to 180 units.

57. Marginal Revenue A manufacturer’s marginal-revenue function is

\[
dr \over dq = 2000 \over \sqrt{300q}
\]

If \( r \) is in dollars, find the change in the manufacturer’s total revenue if production is increased from 500 to 800 units.

58. Marginal Revenue Repeat Problem 57 if

\[
dr \over dq = 100 + 50q - 3q^{2}
\]

and production is increased from 5 to 10 units.

59. Crime Rate A sociologist is studying the crime rate in a certain city. She estimates that \( t \) months after the beginning of the next year, the total number of crimes committed will increase at the rate of \( 8t + 10 \) crimes per month. Determine the total number of crimes that can be expected to be committed next year. How many crimes can be expected to be committed during the last six months of that year?

60. Production Imagine a one-dimensional country of length \( 2R \). (See Figure 13.23.\(^{12}\)) Suppose the production of goods for this country is continuously distributed from border to border. If the amount produced each year per unit of distance is \( f(x) \), then the country’s total yearly production is given by

\[
G = \int_{-R}^{R} f(x) \, dx
\]

Evaluate \( G \) if \( f(x) = i \), where \( i \) is constant.

FIGURE 13.23

61. Exports For the one-dimensional country of Problem 60, under certain conditions the amount of the country’s exports is given by

\[
E = \int_{-R}^{R} \frac{i}{2} [e^{-k(R-x)} + e^{-k(R+x)}] \, dx
\]

where \( i \) and \( k \) are constants (\( k \neq 0 \)). Evaluate \( E \).

62. Average Delivered Price In a discussion of a delivered price of a good from a mill to a customer, DeCanio claims that the average delivered price paid by consumers is given by

\[
A = \int_{0}^{R} \frac{m + x}{[1 - (m + x)]} \, dx
\]

\[
\int_{0}^{R} [1 - (m + x)] \, dx
\]


where \( m \) is mill price, and \( x \) is the maximum distance to the point of sale. DeCanio determines that

\[
A = \frac{\frac{m + R}{2} - m^2 - mR - \frac{R^2}{2}}{1 - m - \frac{R}{2}}
\]

Verify this.

In Problems 63–65, use the Fundamental Theorem of Integral Calculus to determine the value of the definite integral.

13.8 Area between Curves

In Sections 13.6 and 13.7 we saw that the area of a region bounded by the lines \( x = a, x = b, y = 0 \), and a curve \( y = f(x) \) with \( f(x) \geq 0 \) for \( a \leq x \leq b \) can be found by evaluating the definite integral \( \int_a^b f(x) \, dx \). Similarly, for a function \( f(x) \leq 0 \) on an interval \([a, b]\), the area of the region bounded by \( x = a, x = b, y = 0 \), and \( y = f(x) \) is given by \( -\int_a^b f(x) \, dx = \int_a^b (-f(x)) \, dx \). Most of the functions \( f \) we have encountered, and will encounter, are continuous and have a finite number of roots of \( f(x) = 0 \). For such functions, the roots of \( f(x) = 0 \) partition the domain of \( f \) into a finite number of intervals on each of which we have either \( f(x) \geq 0 \) or \( f(x) \leq 0 \). For such a function we can determine the area bounded by \( y = f(x), y = 0 \) and any pair of vertical lines \( x = a \) and \( x = b \), with \( a \) and \( b \) in the domain of \( f \). We have only to find all the roots \( c_1 < c_2 < \cdots < c_k \) with \( a < c_1 \) and \( c_k < b \); calculate the integrals \( \int_{c_1}^{c_2} f(x) \, dx, \int_{c_2}^{c_3} f(x) \, dx, \cdots, \int_{c_k}^{b} f(x) \, dx \); attach to each integral the correct sign to correspond to an area; and add the results. Example 42 will provide a modest example of this idea.

For such an area determination, a rough sketch of the region involved is extremely valuable. To set up the integrals needed, a sample rectangle should be included in the sketch for each individual integral as in Figure 13.24. The area of the region is a limit of sums of areas of rectangles. A sketch helps to understand the integration process and it is indispensable when setting up integrals to find areas of complicated regions. Such a rectangle (see Figure 13.24) is called a vertical strip. In the diagram, the width of the vertical strip is \( \Delta x \). We know from our work on differentials in Section 13.1 that we can consistently write \( \Delta x = dx \), for \( x \) the independent variable. The height of the vertical strip is the \( y \)-value of the curve. Hence, the rectangle has area \( y \Delta x = f(x) \, dx \). The area of the entire region is found by summing the areas of all such vertical strips between \( x = a \) and \( x = b \) and finding the limit of this sum, which is the definite integral. Symbolically, we have

\[
\sum y \Delta x \rightarrow \int_a^b f(x) \, dx
\]

For \( f(x) \geq 0 \) it is helpful to think of \( dx \) as a length differential and \( f(x)dx \) as an area differential \( dA \). Then, as we saw in Section 13.7, we have \( \frac{dA}{dx} = f(x) \) for some area function \( A \) and

\[
\int_a^b f(x) \, dx = \int_a^b dA = A(b) - A(a)
\]
If our area function $A$ measures area starting at the line $x = a$, as it did in Section 13.7, then $A(a) = 0$ and the area under $f$ (and over 0) from $a$ to $b$ is just $A(b)$. It is important to understand here that we need $f(x) \geq 0$ in order to think of $f(x)$ as a length and hence $f(x)dx$ as a differential area. But if $f(x) \leq 0$ then $-f(x) \geq 0$ so that $-f(x)$ becomes a length and $-f(x)dx$ becomes a differential area.

**EXAMPLE 42**  An Area Requiring Two Definite Integrals

Find the area of the region bounded by the curve

$$y = x^2 - x - 2$$

and the line $y = 0$ (the x-axis) from $x = -2$ to $x = 2$.

**Solution:** A sketch of the region is given in Figure 13.24. Notice that the x-intercepts are $(-1, 0)$ and $(2, 0)$.

On the interval $[-2, -1]$, the area of the vertical strip is

$$ydx = (x^2 - x - 2)dx$$

On the interval $[-1, 2]$, the area of the vertical strip is

$$(-y)dx = -(x^2 - x - 2)dx$$

Thus,

$$\text{area} = \int_{-2}^{-1} (x^2 - x - 2) dx + \int_{-1}^{2} -(x^2 - x - 2) dx$$

$$= \left[ \frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_{-2}^{-1} - \left[ \frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_{-1}^{2}$$

$$= \left[ \left( -\frac{1}{3} - \frac{1}{2} + 2 \right) - \left( \frac{8}{3} - \frac{4}{2} + 4 \right) \right]$$

$$- \left[ \left( \frac{8}{3} - \frac{4}{2} - 4 \right) - \left( -\frac{1}{3} - \frac{1}{2} + 2 \right) \right]$$

$$= \frac{19}{3}$$

Now Work Problem 21 ✧

Before embarking on more complicated area problems, we motivate the further study of area by seeing the use of area as a probability in statistics.

**EXAMPLE 43**  Statistics Application

In statistics, a (probability) density function $f$ of a variable $x$, where $x$ assumes all values in the interval $[a, b]$, has the following properties:

(i) $f(x) \geq 0$

(ii) $\int_{a}^{b} f(x) dx = 1$

The probability that $x$ assumes a value between $c$ and $d$, which is written $P(c \leq x \leq d)$, where $a \leq c \leq d \leq b$, is represented by the area of the region bounded by the graph of $f$ and the x-axis between $x = c$ and $x = d$. Hence (see Figure 13.25),

$$P(c \leq x \leq d) = \int_{c}^{d} f(x) dx$$

[In the terminology of Chapters 7 and 8, the condition $c \leq x \leq d$ defines an event and $P(c \leq x \leq d)$ is consistent with the notation of the earlier chapters. Note too that the hypothesis (ii) above ensures that $a \leq x \leq b$ is the certain event.]
For the density function \( f(x) = 6(x - x^2) \), where \( 0 \leq x \leq 1 \), find each of the following probabilities.

a. \( P(0 \leq x \leq \frac{1}{4}) \)

**Solution:** Here \([a, b] = [0, 1] \), \( c = 0 \), and \( d = \frac{1}{4} \). We have

\[
P(0 \leq x \leq \frac{1}{4}) = \int_{0}^{\frac{1}{4}} 6(x - x^2) \, dx = 6 \int_{0}^{\frac{1}{4}} (x - x^2) \, dx
\]

\[
= 6 \left( \frac{x^2}{2} - \frac{x^3}{3} \right) \bigg|_{0}^{\frac{1}{4}} = \left( 3\left(\frac{1}{4}\right)^2 - 2\left(\frac{1}{4}\right)^3 \right) - 0 = \frac{5}{32}
\]

b. \( P(x \geq \frac{1}{2}) \)

**Solution:** Since the domain of \( f \) is \( 0 \leq x \leq 1 \), to say that \( x \geq \frac{1}{2} \) means that \( \frac{1}{2} \leq x \leq 1 \). Thus,

\[
P(x \geq \frac{1}{2}) = \int_{\frac{1}{2}}^{1} 6(x - x^2) \, dx = 6 \int_{\frac{1}{2}}^{1} (x - x^2) \, dx
\]

\[
= 6 \left( \frac{x^2}{2} - \frac{x^3}{3} \right) \bigg|_{\frac{1}{2}}^{1} = \left( 3(1)^2 - 2(1)^3 \right) \bigg|_{\frac{1}{2}}^{1} = \frac{1}{2}
\]

Now Work Problem 27

**EXAMPLE 44 Waiting Time**

While waiting for a bank representative, Siham finds that the waiting time follows an exponential probability density function \( p(x) = 0.4e^{-0.4x} \), where \( x \) the number of minutes before a customer is served. What is the probability that Siham will wait at most five minutes?

**Solution:** At most five minutes means \( 0 \leq t \leq 5 \). The probability that Siham waits at most five minutes is given by

\[
\int_{0}^{5} 0.4e^{-0.4t} \, dt = 0.4 \int_{0}^{5} \frac{1}{-0.4} e^{-0.4t} (-0.4) \, dt
\]

\[
= (-1 \times e^{-0.4t}) \bigg|_{0}^{5} \approx 0.86466
\]
Vertical Strips

We will now find the area of a region enclosed by several curves. As before, our procedure will be to draw a sample strip of area and use the definite integral to “add together” the areas of all such strips.

For example, consider the area of the region in Figure 13.26 that is bounded on the top and bottom by the curves \( y = f(x) \) and \( y = g(x) \) and on the sides by the lines \( x = a \) and \( x = b \). The width of the indicated vertical strip is \( dx \), and the height is the \( y \)-value of the upper curve minus the \( y \)-value of the lower curve, which we will write as \( y_{\text{upper}} - y_{\text{lower}} \). Thus, the area of the strip is

\[
(y_{\text{upper}} - y_{\text{lower}}) \, dx
\]

which is

\[
(f(x) - g(x)) \, dx
\]

Summing the areas of all such strips from \( x = a \) to \( x = b \) by the definite integral gives the area of the region:

\[
\sum (f(x) - g(x)) \, dx \rightarrow \int_a^b (f(x) - g(x)) \, dx = \text{area}
\]

We remark that there is another way to view this area problem. In Figure 13.26 both \( f \) and \( g \) are above \( y = 0 \) and it is clear that the area we seek is also the area under \( f \) minus the area under \( g \). That approach tells us that the required area is

\[
\int_a^b f(x) \, dx - \int_a^b g(x) \, dx = \int_a^b (f(x) - g(x)) \, dx
\]

However, our first approach does not require that either \( f \) or \( g \) lie above 0. Our usage of \( y_{\text{upper}} \) and \( y_{\text{lower}} \) is really just a way of saying that \( f \geq g \) on \([a, b]\). This is equivalent to saying that \( f - g \geq 0 \) on \([a, b]\) so that each differential \((f(x) - g(x)) \, dx\) is meaningful as an area.

**EXAMPLE 45** Finding an Area between Two Curves

Find the area of the region bounded by the curves \( y = \sqrt{x} \) and \( y = x \).

**Solution:** A sketch of the region appears in Figure 13.27. To determine where the curves intersect, we solve the system formed by the equations \( y = \sqrt{x} \) and \( y = x \). Eliminating \( y \) by substitution, we obtain

\[
\sqrt{x} = x \\
x = x^2
\]

squaring both sides

\[
0 = x^2 - x = x(x - 1)
\]

\[
x = 0 \quad \text{or} \quad x = 1
\]
Since we squared both sides, we must check the solutions found with respect to the original equation. It is easily determined that both \( x = 0 \) and \( x = 1 \) are solutions of \( \sqrt{x} = x \). If \( x = 0 \), then \( y = 0 \); if \( x = 1 \), then \( y = 1 \). Thus, the curves intersect at \((0,0)\) and \((1,1)\). The width of the indicated strip of area is \( dx \). The height is the \( y \)-value on the upper curve minus the \( y \)-value on the lower curve:

\[
y_{\text{upper}} - y_{\text{lower}} = \sqrt{x} - x
\]

Hence, the area of the strip is \((\sqrt{x} - x)dx\). Summing the areas of all such strips from \( x = 0 \) to \( x = 1 \) by the definite integral, we get the area of the entire region:

\[
\text{area} = \int_{0}^{1} (\sqrt{x} - x) \, dx
\]

\[
= \int_{0}^{1} (\sqrt{x} - x) \, dx = \left( \frac{x^{3/2}}{\frac{3}{2}} - \frac{x^2}{2} \right) \bigg|_{0}^{1}
\]

\[
= \left( \frac{2}{3} - \frac{1}{2} \right) - (0 - 0) = \frac{1}{6}
\]

Now Work Problem 45

**EXAMPLE 46** Finding an Area between Two Curves

Find the area of the region bounded by the curves \( y = 4x - x^2 + 8 \) and \( y = x^2 - 2x \).

**Solution:** A sketch of the region appears in Figure 13.28. To find where the curves intersect, we solve the system of equations \( y = 4x - x^2 + 8 \) and \( y = x^2 - 2x \):

\[
4x - x^2 + 8 = x^2 - 2x,
\]

\[-2x^2 + 6x + 8 = 0,
\]

\[x^2 - 3x - 4 = 0,
\]

\[(x + 1)(x - 4) = 0 \quad \text{factoring}
\]

\[x = -1 \quad \text{or} \quad x = 4
\]

When \( x = -1 \), then \( y = 3 \); when \( x = 4 \), then \( y = 8 \). Thus, the curves intersect at \((-1,3)\) and \((4,8)\). The width of the indicated strip is \( dx \). The height is the \( y \)-value on the upper curve minus the \( y \)-value on the lower curve:

\[
y_{\text{upper}} - y_{\text{lower}} = (4x - x^2 + 8) - (x^2 - 2x)
\]

Therefore, the area of the strip is

\[
[(4x - x^2 + 8) - (x^2 - 2x)] \, dx = (-2x^2 + 6x + 8) \, dx
\]

Summing all such areas from \( x = -1 \) to \( x = 4 \), we have

\[
\text{area} = \int_{-1}^{4} (-2x^2 + 6x + 8) \, dx = 41\frac{2}{3}
\]

Now Work Problem 49

**EXAMPLE 47** Area of a Region Having Two Different Upper Curves

Find the area of the region between the curves \( y = 9 - x^2 \) and \( y = x^2 + 1 \) from \( x = 0 \) to \( x = 3 \).

**Solution:** The region is sketched in Figure 13.29. The curves intersect when

\[
9 - x^2 = x^2 + 1
\]

\[8 = 2x^2
\]

\[4 = x^2
\]

\[x = \pm2 \quad \text{two solutions}
\]
When \( x = \pm 2 \), then \( y = 5 \), so the points of intersection are \((\pm 2, 5)\). Because we are interested in the region from \( x = 0 \) to \( x = 3 \), the intersection point that is of concern to us is \((2, 5)\). Notice in Figure 13.29 that in the region to the left of the intersection point \((2, 5)\), a strip has

\[
y_{\text{upper}} = 9 - x^2 \quad \text{and} \quad y_{\text{lower}} = x^2 + 1
\]

but for a strip to the right of \((2, 5)\) the reverse is true, namely,

\[
y_{\text{upper}} = x^2 + 1 \quad \text{and} \quad y_{\text{lower}} = 9 - x^2
\]

Thus, from \( x = 0 \) to \( x = 2 \), the area of a strip is

\[
(y_{\text{upper}} - y_{\text{lower}}) \, dx = [(9 - x^2) - (x^2 + 1)] \, dx
= (8 - 2x^2) \, dx
\]

but from \( x = 2 \) to \( x = 3 \), it is

\[
(y_{\text{upper}} - y_{\text{lower}}) \, dx = [(x^2 + 1) - (9 - x^2)] \, dx
= (2x^2 - 8) \, dx
\]

Therefore, to find the area of the entire region, we need two integrals:

\[
\text{area} = \int_{0}^{2} (8 - 2x^2) \, dx + \int_{2}^{3} (2x^2 - 8) \, dx
= \left[ (8x - \frac{2}{3}x^3) \right]_{0}^{2} + \left[ (\frac{2}{3}x^3 - 8x) \right]_{2}^{3}
= \left[ (16 - \frac{16}{3}) - 0 \right] + \left[ (18 - 24) - (\frac{16}{3} - 16) \right]
= \frac{46}{3}
\]

Now Work Problem 41

**Horizontal Strips**

Sometimes area can more easily be determined by summing areas of horizontal strips rather than vertical strips. In the following example, an area will be found by both methods. In each case, the strip of area determines the form of the integral.

**EXAMPLE 48 Vertical Strips and Horizontal Strips**

Find the area of the region bounded by the curve \( y^2 = 4x \) and the lines \( y = 3 \) and \( x = 0 \) (the y-axis).

**Solution:** The region is sketched in Figure 13.30. When the curves \( y = 3 \) and \( y^2 = 4x \) intersect, \( 9 = 4x \), so \( x = \frac{9}{4} \). Thus, the intersection point is \((\frac{9}{4}, 3)\). Since the width of the vertical strip is \( dx \), we integrate with respect to the variable \( x \). Accordingly, \( y_{\text{upper}} \) and \( y_{\text{lower}} \) must be expressed as functions of \( x \). For the lower curve, \( y^2 = 4x \), we have \( y = \pm \sqrt{4x} \). But \( y \geq 0 \) for the portion of this curve that bounds the region, so we use \( y = 2\sqrt{x} \). The upper curve is \( y = 3 \). Hence, the height of the strip is

\[
y_{\text{upper}} - y_{\text{lower}} = 3 - 2\sqrt{x}
\]

Therefore, the strip has an area of \((3 - 2\sqrt{x}) \, dx\), and we wish to sum all such areas from \( x = 0 \) to \( x = \frac{9}{4} \). We have

\[
\text{area} = \int_{0}^{9/4} (3 - 2\sqrt{x}) \, dx
= \left( 3x - \frac{4x^{3/2}}{3} \right) \bigg|_{0}^{9/4}
\]
Let us now approach this problem from the point of view of a horizontal strip as shown in Figure 13.31. The width of the strip is \( dy \). The length of the strip is the \( x \)-value on the rightmost curve minus the \( x \)-value on the leftmost curve. Thus, the area of the strip is

\[
\text{area} = (x_{\text{right}} - x_{\text{left}}) dy
\]

We wish to sum all such areas from \( y = 0 \) to \( y = 3 \):

\[
\sum (x_{\text{right}} - x_{\text{left}}) dy \rightarrow \int_{0}^{3} (x_{\text{right}} - x_{\text{left}}) dy
\]

Since the variable of integration is \( y \), we must express \( x_{\text{right}} \) and \( x_{\text{left}} \) as functions of \( y \). The rightmost curve is \( y^2 = 4x \) so that \( x = y^2/4 \). The left curve is \( x = 0 \). Thus,

\[
\text{area} = \int_{0}^{3} (x_{\text{right}} - x_{\text{left}}) dy = \int_{0}^{3} \left( \frac{y^2}{4} - 0 \right) dy = \frac{y^3}{12} \bigg|_{0}^{3} = \frac{9}{4}
\]

Note that for this region, horizontal strips make the definite integral easier to evaluate (and set up) than an integral with vertical strips. In any case, remember that the limits of integration are limits for the variable of integration.

**Now Work Problem 53**

---

**EXAMPLE 49**  Advantage of Horizontal Elements

Find the area of the region bounded by the graphs of \( y^2 = x \) and \( x - y = 2 \).

**Solution:** The region is sketched in Figure 13.32. The curves intersect when \( y^2 - y = 2 \). Thus, \( y^2 - y - 2 = 0 \); equivalently, \( (y + 1)(y - 2) = 0 \), from which it follows that \( y = -1 \) or \( y = 2 \). This gives the intersection points \((1, -1)\) and \((4, 2)\).

Let us try vertical strips of area. [See Figure 13.32(a).] Solving \( y^2 = x \) for \( y \) gives \( y = \pm \sqrt{x} \). As seen in Figure 13.32(a), to the left of \( x = 1 \), the upper end of the strip lies on \( y = \sqrt{x} \) and the lower end lies on \( y = -\sqrt{x} \). To the right of \( x = 1 \), the upper curve is \( y = \sqrt{x} \) and the lower curve is \( x - y = 2 \) (or \( y = x - 2 \)). Thus, with vertical strips, two integrals are needed to evaluate the area:

\[
\text{area} = \int_{0}^{1} (\sqrt{x} - (-\sqrt{x})) dx + \int_{1}^{4} (\sqrt{x} - (x - 2)) dx
\]

---

**FIGURE 13.32** Region of Example 49 with vertical and horizontal strips.
Perhaps the use of horizontal strips can simplify our work. In Figure 13.32(b), the width
of the strip is \( \Delta y \). The rightmost curve is always \( x - y = 2 \) (or \( x = y + 2 \)), and
the leftmost curve is always \( y^2 = x \) (or \( x = y^2 \)). Therefore, the area of the horizontal
strip is \( (y + 2 - y^2) \Delta y \), so the total area is

\[
\text{area} = \int_{-1}^{2} (y + 2 - y^2) \, dy = \frac{9}{2}
\]

Clearly, the use of horizontal strips is the most desirable approach to solving the prob-
lem. Only a single integral is needed, and it is much simpler to compute.

Now Work Problem 55

PROBLEMS 13.8

In Problems 1–23, use a definite integral to find the area of the
region bounded by the given curve, the x-axis, and the given lines.
In each case, first sketch the region. Watch out for areas of regions
that are below the x-axis.

1. \( y = 5x + 2 \), \( x = 1 \), \( x = 4 \)
2. \( y = x + 5 \), \( x = 2 \), \( x = 4 \)
3. \( y = 3x^2 \), \( x = 1 \), \( x = 3 \)
4. \( y = x^2 \), \( x = 2 \), \( x = 3 \)
5. \( y = x + x^2 + x^3 \), \( x = 1 \)
6. \( y = x^2 - 2x \), \( x = -3 \), \( x = -1 \)
7. \( y = 2 - x - x^2 \)
8. \( y = \frac{4}{x} \), \( x = 1 \), \( x = 2 \)
9. \( y = 2 - x - x^2 \), \( x = -3 \), \( x = 0 \)
10. \( y = e^x \), \( x = 1 \), \( x = 3 \)
11. \( y = \frac{1}{(x - 1)^2} \), \( x = 2 \), \( x = 3 \)
12. \( y = \frac{1}{x} \), \( x = 1 \), \( x = e \)
13. \( y = \sqrt{x + 9} \), \( x = -9 \), \( x = 0 \)
14. \( y = x^2 - 4x \), \( x = 2 \), \( x = 6 \)
15. \( y = \sqrt{2x - 1} \), \( x = 1 \), \( x = 5 \)
16. \( y = x^3 + 3x^2 \), \( x = -2 \), \( x = 2 \)
17. \( y = \sqrt{x} \), \( x = 2 \)
18. \( y = e^x + 1 \), \( x = 0 \), \( x = 1 \)
19. \( y = |x| \), \( x = -2 \), \( x = 2 \)
20. \( y = x + \frac{2}{x} \), \( x = -1 \), \( x = 2 \)
21. \( y = x^3 \), \( x = -2 \), \( x = 4 \)
22. \( y = \sqrt{x - 2} \), \( x = 2 \), \( x = 6 \)
23. \( y = x^3 + 1 \), \( x = 0 \), \( x = 4 \)
24. Given that

\[
f(x) = \begin{cases} 
3x^2 & \text{if } 0 \leq x < 2 \\
16 - 2x & \text{if } x \geq 2 
\end{cases}
\]

determine the area of the region bounded by the graph of
\( y = f(x) \), the x-axis, and the line \( x = 3 \). Include a sketch
of the region.

25. Under conditions of a continuous uniform distribution (a
topic in statistics), the proportion of persons with incomes
between \( a \) and \( t \), where \( a \leq t \leq b \), is the area of the region
between the curve \( y = 1/(b - a) \) and the x-axis from \( x = a \) to
\( x = t \). Sketch the graph of the curve and determine the area
of the given region.

26. Suppose \( f(x) = \frac{1}{2}(1 - x^2) \), where \( 0 \leq x \leq 3 \). If \( f \) is a density
function (refer to Example 43), find each of the following.
(a) \( P(1 \leq x \leq 2) \)
(b) \( P(1 \leq x \leq \frac{3}{2}) \)
(c) \( P(x \leq 1) \)
(d) \( P(x \geq 1) \) using your result from part (c)

27. Suppose \( f(x) = x/8 \), where \( 0 \leq x \leq 4 \). If \( f \) is a density
function (refer to Example 43), find each of the following.
(a) \( P(0 \leq x \leq 1) \)
(b) \( P(2 \leq x \leq 4) \)
(c) \( P(x \geq 3) \)

28. Suppose \( f(x) = 1/x \), where \( 0 \leq x \leq e^2 \). If \( f \) is a density
function (refer to Example 43), find each of the following.
(a) \( P(3 \leq x \leq 7) \)
(b) \( P(x \leq 5) \)
(c) \( P(x \geq 4) \)
(d) Verify that \( P(e \leq x \leq e^2) = 1 \).

29. (a) Let \( r \) be a real number, where \( r > 1 \). Evaluate

\[
\int_{1}^{r} \frac{1}{x^2} \, dx
\]

(b) Your answer to part (a) can be interpreted as the area of a
certain region of the plane. Sketch this region.
(c) Evaluate \( \lim_{r \to \infty} \left( \int_{1}^{r} \frac{1}{x^2} \, dx \right) \).
(d) Your answer to part (c) can be interpreted as the area of a
certain region of the plane. Sketch this region.

In Problems 30–33, use definite integration to estimate the area
of the region bounded by the given curve, the x-axis, and the
given lines. Round your answer to two decimal places.

30. \( y = \frac{1}{x^2 + 1} \), \( x = -2 \), \( x = 1 \)
In Problems 34–37, express the area of the shaded region in terms of an integral (or integrals). Do not evaluate your expression.

34. See Figure 13.33.

35. See Figure 13.34.

36. See Figure 13.35.

37. See Figure 13.36.

38. Express, in terms of a single integral, the total area of the region to the right of the line \( x = 1 \) that is between the curves \( y = x^2 - 5 \) and \( y = 7 - 2x^2 \). Do not evaluate the integral.

In Problems 39–54, find the area of the region bounded by the graphs of the given equations. Be sure to find any needed points of intersection. Consider whether the use of horizontal strips makes the integral simpler than when vertical strips are used.

39. \( y = x^2 \), \( y = 2x \)
40. \( y = 10 - x^2 \), \( y = 4 \)
41. \( y = x \), \( y = -x + 3 \), \( y = 0 \)
42. \( y^2 = x + 1 \), \( x = 1 \)
43. \( x = 8 + 2y \), \( x = 0 \), \( y = -1 \), \( y = 3 \)
44. \( y = x - 6 \), \( y^2 = x \)
45. \( y^2 = 4x \), \( y = 2x - 4 \)
46. \( y = x^2 \), \( y = x + 6 \), \( x = 0 \).
(Hint: The only real root of \( x^3 - x - 6 = 0 \) is 2.)
47. \( 2y = 4x - x^2 \), \( 2y = x - 4 \)
48. \( y = \sqrt{x} \), \( y = x^2 \)
49. \( y = 8 - x^2 \), \( y = x^2 \), \( x = -1 \), \( x = 1 \)
50. \( y = x^3 + x, y = 0, x = -1, x = 2 \)
51. \( y = x^3 - 1 \), \( y = x - 1 \)
52. \( y = x^3 \), \( y = \sqrt{x} \)
53. \( y^2 = -x - 2 \), \( x - y = 5 \), \( y = -1 \), \( y = 1 \)
54. \( 4x + 4y + 17 = 0 \), \( y = \frac{1}{x} \)

55. Find the area of the region that is between the curves

\[ y = x - 1 \quad \text{and} \quad y = 5 - 2x \]

from \( x = 0 \) to \( x = 4 \).

56. Find the area of the region that is between the curves

\[ y = x^2 - 4x + 4 \quad \text{and} \quad y = 10 - x^2 \]

from \( x = 2 \) to \( x = 4 \).

57. **Lorenz Curve** A Lorenz curve is used in studying income distributions. If \( x \) is the cumulative percentage of income recipients, ranked from poorest to richest, and \( y \) is the cumulative percentage of income, then equality of income distribution is given by the line \( y = x \) in Figure 13.37, where \( x \) and \( y \) are expressed as decimals. For example, 10% of the people receive 10% of total income, 20% of the people receive 20% of the
income, and so on. Suppose the actual distribution is given by the Lorenz curve defined by

\[ y = \frac{14}{15}x^2 + \frac{1}{15}x \]

Note, for example, that 30% of the people receive only 10.4% of total income. The degree of deviation from equality is measured by the coefficient of inequality for a Lorenz curve. This coefficient is defined to be the area between the curve and the diagonal, divided by the area under the diagonal:

\[
\frac{\text{area between curve and diagonal}}{\text{area under diagonal}}
\]

For example, when all incomes are equal, the coefficient of inequality is zero. Find the coefficient of inequality for the Lorenz curve just defined.

58. Lorenz curve Find the coefficient of inequality as in Problem 57 for the Lorenz curve defined by \( y = \frac{14}{15}x^2 + \frac{1}{15}x \).

59. Find the area of the region bounded by the graphs of the equations \( y^2 = 3x \) and \( y = mx \), where \( m \) is a positive constant.

60. (a) Find the area of the region bounded by the graphs of \( y = x^2 - 1 \) and \( y = 2x + 2 \).
   (b) What percentage of the area in part (a) lies above the x-axis?

61. The region bounded by the curve \( y = x^2 \) and the line \( y = 4 \) is divided into two parts of equal area by the line \( y = k \), where \( k \) is a constant. Find the value of \( k \).

In Problems 62–66, estimate the area of the region bounded by the graphs of the given equations. Round your answer to two decimal places.

62. \( y = x^2 - 4x + 1, \quad y = -\frac{6}{x} \)

63. \( y = \sqrt{25 - x^2}, \quad y = 7 - 2x - x^4 \)

64. \( y = x^2 - 8x + 1, \quad y = x^2 - 5 \)

65. \( y = x^2 - 3x^3 + 2x, \quad y = 3x^2 - 4 \)

66. \( y = x^2 - 3x^3 - 15x^2 + 19x + 30, \quad y = x^3 + x^2 - 20x \)

13.9 Consumers’ and Producers’ Surplus

Determining the area of a region has applications in economics. Figure 13.38 shows a supply curve for a product. The curve indicates the price \( p \) per unit at which the manufacturer will sell (or supply) \( q \) units. The diagram also shows a demand curve for the product. This curve indicates the price \( p \) per unit at which consumers will purchase (or demand) \( q \) units. The point \((q_0, p_0)\) where the two curves intersect is called the point of equilibrium. Here \( p_0 \) is the price per unit at which consumers will purchase the same quantity \( q_0 \) of a product that producers wish to sell at that price. In short, \( p_0 \) is the price at which stability in the producer–consumer relationship occurs.

Let us assume that the market is at equilibrium and the price per unit of the product is \( p_0 \). According to the demand curve, there are consumers who would be willing to pay more than \( p_0 \). For example, at the price per unit of \( p_1 \), consumers would buy \( q_1 \) units. These consumers are benefiting from the lower equilibrium price \( p_0 \).

The vertical strip in Figure 13.38 has area \( p dq \). This expression can also be thought of as the total amount of money that consumers would spend by buying \( dq \) units of the product if the price per unit were \( p \). Since the price is actually \( p_0 \), these consumers spend only \( p_0 dq \) for the \( dq \) units and thus benefit by the amount \( pdq - p_0 dq \). This expression can be written \( (p - p_0) dq \), which is the area of a rectangle of width \( dq \) and height \( p - p_0 \). (See Figure 13.39.) Summing the areas of all such rectangles from \( q = 0 \) to \( q = q_0 \) by definite integration, we have

\[
\int_0^{q_0} (p - p_0) dq
\]

This integral, under certain conditions, represents the total gain to consumers who are willing to pay more than the equilibrium price. This total gain is called consumers’ surplus, abbreviated CS. If the demand function is given by \( p = f(q) \), then

\[
CS = \int_0^{q_0} [f(q) - p_0] \, dq
\]

Geometrically (see Figure 13.40), consumers’ surplus is represented by the area between the line \( p = p_0 \) and the demand curve \( p = f(q) \) from \( q = 0 \) to \( q = q_0 \).

Some of the producers also benefit from the equilibrium price, since they are willing to supply the product at prices less than \( p_0 \). Under certain conditions, the total gain to the producers is represented geometrically in Figure 13.41 by the area between the line \( p = p_0 \) and the supply curve \( p = g(q) \) from \( q = 0 \) to \( q = q_0 \). This gain, called producers’ surplus and abbreviated PS, is given by

\[
PS = \int_0^{q_0} [p_0 - g(q)] \, dq
\]

**EXAMPLE 50 Producers’ Surplus**

The manager of Ibn Al-Haitham Sunglasses store has found that the demand equation for the most popular glasses is given by \( p = \sqrt{25 + 3q} \), where \( p \) is the price in hundreds of dollars, at which \( q \) pairs of glasses are sold. If the market price is $14, what is the producers’ surplus?

**Solution:** When \( p = 14 \) we have \( 14 = \sqrt{25 + 3q} \), from which we find that \( q = 57 \).

The producers’ surplus is given by

\[
PS = \int_0^{q_0} (p_0 - p) \, dq = \int_0^{57} (14 - \sqrt{25 + 3q}) \, dq
\]

\[
= \int_0^{57} 14 \, dq - \int_0^{57} \sqrt{25 + 3q} \, dq
\]

\[
= 14q - \frac{1}{3} (25 + 3q)^{3/2} \bigg|_0^{57}
\]

\[
= 14q - \frac{2}{9} (25 + 3q)^{3/2} \approx 216
\]

Therefore the producers’ surplus is approximately $21,600.

**EXAMPLE 51 Finding Consumers’ Surplus and Producers’ Surplus**

The demand function for a product is

\[ p = f(q) = 100 - 0.05q \]

where \( p \) is the price per unit (in dollars) for \( q \) units. The supply function is

\[ p = g(q) = 10 + 0.1q \]

Determine consumers’ surplus and producers’ surplus under market equilibrium.

**Solution:** First we must find the equilibrium point \((p_0, q_0)\) by solving the system formed by the functions \( p = 100 - 0.05q \) and \( p = 10 + 0.1q \). We thus equate the two expressions for \( p \) and solve:

\[ 10 + 0.1q = 100 - 0.05q \]

\[ 0.15q = 90 \]

\[ q = 600 \]
When \( q = 600 \) then \( p = 10 + 0.1(600) = 70 \). Hence, \( q_0 = 600 \) and \( p_0 = 70 \). Consumers’ surplus is

\[
CS = \int_0^{600} [f(q) - p_0] \, dq = \int_0^{600} (100 - 0.05q - 70) \, dq = \left( 30q - 0.05q^2 \right) \bigg|_0^{600} = 9000
\]

Producers’ surplus is

\[
PS = \int_0^{600} [p_0 - g(q)] \, dq = \int_0^{600} [70 - (10 + 0.1q)] \, dq = \left( 60q - 0.1q^2 \right) \bigg|_0^{600} = 18,000
\]

Therefore, consumers’ surplus is $9000 and producers’ surplus is $18,000.

**EXAMPLE 52** Using Horizontal Strips to Find Consumers’ Surplus and Producers’ Surplus

The demand equation for a product is

\[
q = f(p) = \frac{90}{p} - 2
\]

and the supply equation is \( q = g(p) = p - 1 \). Determine consumers’ surplus and producers’ surplus when market equilibrium has been established.

**Solution:** Determining the equilibrium point, we have

\[
p - 1 = \frac{90}{p} - 2
\]

\[
p^2 + p - 90 = 0
\]

\[
(p + 10)(p - 9) = 0
\]

Thus, \( p_0 = 9 \), so \( q_0 = 9 - 1 = 8 \). (See Figure 13.42.) Note that the demand equation expresses \( q \) as a function of \( p \). Since consumers’ surplus can be considered an area, this area can be determined by means of horizontal strips of width \( dp \) and length \( q = f(p) \). The areas of these strips are summed from \( p = 9 \) to \( p = 45 \) by integrating with respect to \( p \):

![Diagram for Example 52](image)
\[ \text{CS} = \int_{9}^{45} \left( \frac{90}{p} - 2 \right) \, dp = (90 \ln |p| - 2p) \bigg|_{9}^{45} \\
= 90 \ln 5 - 72 \approx 72.85 \]

Using horizontal strips for producers’ surplus, we have
\[ \text{PS} = \int_{1}^{9} (p - 1) \, dp = \left( \frac{(p - 1)^2}{2} \right) \bigg|_{1}^{9} = 32 \]

Now Work Problem 5<1

PROBLEMS 13.9

In Problems 1–5, the first equation is a demand equation and the second is a supply equation of a product. In each case, determine consumers’ surplus and producers’ surplus under market equilibrium.

1. \[ p = 22 - 0.8q \quad \text{and} \quad p = 6 + 1.2q \]
2. \[ p = 2200 - q^2 \quad \text{and} \quad p = 400 + q^2 \]
3. \[ p = 900 - q^2 \quad \text{and} \quad p = 10q + 300 \]
4. \[ q = \sqrt{100 - p} \quad \text{and} \quad q = \frac{p}{2} - 10 \]
5. \[ q = 100(10 - 2p) \quad \text{and} \quad q = 50(2p - 1) \]
6. The demand equation for a product is \[ q = 10\sqrt{100 - p} \]
Calculate consumers’ surplus under market equilibrium, which occurs at a price of $84.

7. The demand equation for a product is \[ q = 400 - p^2 \]
and the supply equation is \[ p = \frac{q}{60} + 5 \]

8. The demand equation for a product is \[ p = 2^{10 - q} \]
and the supply equation is \[ p = 2^{q^2} \]
where \( p \) is the price per unit (in hundreds of dollars) when \( q \) units are demanded or supplied. Determine, to the nearest thousand dollars, consumers’ surplus under market equilibrium.

9. The demand equation for a product is \[ p = 60 - \frac{50q}{\sqrt{q^2 + 3600}} \]
and the supply equation is \[ p = 10 \ln (q + 20) - 26 \]
Determine consumers’ surplus and producers’ surplus under market equilibrium. Round your answers to the nearest integer.

10. Producers’ Surplus

The supply function for a product is given by the following table, where \( p \) is the price per unit (in dollars) at which \( q \) units are supplied to the market:

<table>
<thead>
<tr>
<th>( q )</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>25</td>
<td>49</td>
<td>59</td>
<td>71</td>
<td>80</td>
<td>94</td>
</tr>
</tbody>
</table>

Estimate the producers’ surplus if the selling price is $80.

Chapter 13 Review

Important Terms and Symbols

| Section 13.1 | Differentials | differential, \( dy, dx \) | Ex. 1, p. 672 |
| Section 13.2 | The Indefinite Integral | antiderivative, indefinite integral \( \int f(x) \, dx \), integral sign \( \int \), constant of integration \( c \) | Ex. 6, p. 678 |
| Section 13.3 | Integration with Initial Conditions | initial condition \( f(a) \) | Ex. 15, p. 683 |
| Section 13.4 | More Integration Formulas | power rule for integration \( (x^n) \) | Ex. 21, p. 688 |
| Section 13.5 | Techniques of Integration | preliminary division \( \frac{1}{x^a} \) | Ex. 29, p. 694 |
| Section 13.6 | The Definite Integral | definite integral \( \int_a^b f(x) \, dx \), limits of integration \( a \) and \( b \) | Ex. 34, p. 703 |
Section 13.7 The Fundamental Theorem of Integral Calculus

Fundamental Theorem of Integral Calculus \( F(x) \bigg|_a^b \)

Ex. 36, p. 707

Section 13.8 Area between Curves

vertical strip, horizontal strip

Ex. 42, p. 715

Ex. 48, p. 719

Section 13.9 Consumers’ and Producers’ Surplus

consumers’ surplus, producers’ surplus

Ex. 51, p. 724

Summary

If \( y = f(x) \) is a differentiable function of \( x \), we define the differential \( dy \) by

\[
dy = f'(x) \, dx
\]

where \( dx = \Delta x \) is a change in \( x \) and can be any real number. (Thus \( dy \) is a function of two variables, namely \( x \) and \( dx \).) If \( dx \) is close to zero, then \( dy \) is an approximation to \( \Delta y = f(x + dx) - f(x) \).

\[ \Delta y \approx dy \]

Moreover, \( dy \) can be used to approximate a function value using

\[ f(x + dx) \approx f(x) + dy \]

An antiderivative of a function \( f \) is a function \( F \) such that \( F'(x) = f(x) \). Any two antiderivatives of \( f \) differ at most by a constant. The most general antiderivative of \( f \) is called the indefinite integral of \( f \) and is denoted \( \int f(x) \, dx \). Thus,

\[ \int f(x) \, dx = F(x) + C \]

where \( C \) is called the constant of integration, if and only if \( F' = f \).

Some elementary integration formulas are as follows:

\[
\begin{align*}
\int k \, dx &= kx + C \quad k \text{ a constant} \\
\int x^a \, dx &= \frac{x^{a+1}}{a+1} + C \quad a \neq -1 \\
\int \frac{1}{x} \, dx &= \ln |x| + C \quad \text{for } x > 0 \\
\int e^x \, dx &= e^x + C \\
\int kf(x) \, dx &= k \int f(x) \, dx \quad k \text{ a constant} \\
\int [f(x) \pm g(x)] \, dx &= \int f(x) \, dx \pm \int g(x) \, dx
\end{align*}
\]

Another formula is the power rule for integration:

\[ \int u^a \, du = \frac{u^{a+1}}{a+1} + C, \quad \text{if } a \neq -1 \]

Here \( u \) represents a differentiable function of \( x \), and \( du \) is its differential. In applying the power rule to a given integral, it is important that the integral be written in a form that precisely matches the power rule. Other integration formulas are

\[
\begin{align*}
\int e^x \, du &= e^x + C \\
\int \frac{1}{u} \, du &= \ln |u| + C \quad u \neq 0
\end{align*}
\]

If the rate of change of a function \( f \) is known—that is, if \( f' \) is known—then \( f \) is an antiderivative of \( f' \). In addition, if we know that \( f \) satisfies an initial condition, then we can find the particular antiderivative. For example, if a marginal-cost function \( dc/dq \) is given to us, then by integration, we can find the most general form of \( c \). That form involves a constant of integration. However, if we are also given fixed costs (that is, costs involved when \( q = 0 \)), then we can determine the value of the constant of integration and thus find the particular cost function \( c \). Similarly, if we are given a marginal-revenue function \( dr/dq \), then by integration and by using the fact that \( r = 0 \) when \( q = 0 \), we can determine the particular revenue function \( r \). Once \( r \) is known, the corresponding demand equation can be found by using the equation \( p = r/q \).

It is helpful at this point to review summation notation from Section 4.1. This notation is especially useful in determining areas. For continuous \( f \geq 0 \), to find the area of the region bounded by \( y = f(x), y = 0, x = a, \) and \( x = b, \) we divide the interval \([a, b]\) into \( n \) subintervals of equal length \( dx = (b-a)/n \). If \( x_i \) is the right-hand endpoint of an arbitrary subinterval, then the product \( f(x_i) \, dx \) is the area of a rectangle. Denoting the sum of all such areas of rectangles for the \( n \) subintervals by \( S_n \), we define the limit of \( S_n \) as \( n \to \infty \) as the area of the entire region:

\[
\lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \, dx = \text{area}
\]

If the restriction that \( f(x) \geq 0 \) is omitted, this limit is defined as the definite integral of \( f \) over \([a, b]\):

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \, dx = \int_{a}^{b} f(x) \, dx
\]

Instead of evaluating definite integrals by using limits, we may be able to employ the Fundamental Theorem of Integral Calculus. Mathematically,

\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) \]

where \( F \) is any antiderivative of \( f \).
Some properties of the definite integral are

\[ \int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx \quad k \text{ a constant} \]

\[ \int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx \]

and

\[ \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \]

If \( f(x) \geq 0 \) is continuous on \([a, b]\), then the definite integral can be used to find the area of the region bounded by \( y = f(x) \), the x-axis, \( x = a \), and \( x = b \). The definite integral can also be used to find areas of more complicated regions. In these situations, a strip of area should be drawn in the region. This allows us to set up the proper definite integral. In this regard, both vertical strips and horizontal strips have their uses.

One application of finding areas involves consumers’ surplus and producers’ surplus. Suppose the market for a product is at equilibrium and \((q_0, p_0)\) is the equilibrium point (the point of intersection of the supply curve and the demand curve for the product). Then consumers’ surplus, \(CS\), corresponds to the area from \( q = 0 \) to \( q = q_0 \), bounded above by the demand curve and below by the line \( p = p_0 \). Thus,

\[ CS = \int_0^{q_0} (f(q) - p_0) \, dq \]

where \( f \) is the demand function. Producers’ surplus, \(PS\), corresponds to the area from \( q = 0 \) to \( q = q_0 \), bounded above by the line \( p = p_0 \) and below by the supply curve. Therefore,

\[ PS = \int_0^{q_0} (p_0 - g(q)) \, dq \]

where \( g \) is the supply function.

**Review Problems**

**In Problems 1–40, determine the integrals.**

1. \( \int (x^3 + 2x - 7) \, dx \)  
2. \( \int dx \)  
3. \( \int_0^{12} (9\sqrt{3x} + 3x^2) \, dx \)  
4. \( \int \frac{4}{5 - 3x} \, dx \)  
5. \( \int \frac{6}{(x + 5)^2} \, dx \)  
6. \( \int_3^9 (y - 6)^3 \, dy \)  
7. \( \int \frac{6x^3 - 12}{x^3 - 6x + 1} \, dx \)  
8. \( \int_0^3 2xe^{x^2} \, dx \)  
9. \( \int_0^1 \sqrt[5]{3x + 8} \, dt \)  
10. \( \int_0^1 \frac{4 - 2x}{7} \, dx \)  
11. \( \int y(y + 1)^2 \, dy \)  
12. \( \int_0^1 10^{-8} \, dx \)  
13. \( \int \sqrt{t} - \sqrt{i} \, dt \)  
14. \( \int \frac{(0.5x - 0.1)^4}{0.4} \, dx \)  
15. \( \int_1^3 \frac{2t^2}{3 + 2t^3} \, dt \)  
16. \( \int \frac{4x^2 - x}{x} \, dx \)  
17. \( \int x^2 \sqrt{3x + 2} \, dx \)  
18. \( \int (6x^2 + 4x)(x^3 + x^2)^{1/2} \, dx \)  
19. \( \int (e^{2x} - e^{-2y}) \, dy \)  
20. \( \int \frac{8x}{3\sqrt[3]{7 - 2x^2}} \, dx \)  
21. \( \int \frac{\left( \frac{1}{x} + \frac{2}{x^2} \right)}{x} \, dx \)  
22. \( \int_0^2 \frac{3e^{3x}}{1 + e^{3x}} \, dx \)  
23. \( \int_2^7 (y^2 + y^3 + y^2 + y) \, dy \)  
24. \( \int_1^{70} dx \)  
25. \( \int_1^5 5x \sqrt{5 - x^2} \, dx \)  
26. \( \int_0^1 (2x + 1)(x^3 + x)^2 \, dx \)  
27. \( \int_0^1 \left[ 2x - \frac{1}{(x + 1)^{2/3}} \right] \, dx \)  
28. \( \int_0^{18} (2x - 3\sqrt[3]{2x} + 1) \, dx \)  
29. \( \int \frac{\sqrt{7} - 3}{t^2} \, dt \)  
30. \( \int \frac{3z^3}{z - 1} \, dz \)  
31. \( \int_{-1}^0 \frac{x^2 + 4x - 1}{x + 2} \, dx \)  
32. \( \int \frac{(x^2 + 4)^2}{x^2} \, dx \)  
33. \( \int \frac{e^{2x} + x}{2\sqrt{x}} \, dx \)  
34. \( \int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx \)  
35. \( \int e^{ln(x)} \, dx \)  
36. \( \int \frac{6x^2 + 4}{e^{x^2} + 2x} \, dx \)  
37. \( \int \frac{(1 + e^{2y})}{e^{-2y}} \, dx \)  
38. \( \int \frac{e^{a(x + e^{-bx})}}{n \neq 1} \, dx \)  
39. \( \int 3\sqrt{10x} \, dx \)  
40. \( \int \frac{5x^3 + 15x^2 + 37x + 3}{x^2 + 3x + 7} \, dx \)

In Problems 41 and 42, find \( y \), subject to the given condition.

41. \( y' = e^{x^3} + 3 \), \( y(0) = -\frac{1}{2} \)  
42. \( y' = \frac{x + 5}{x}, \quad y(1) = 3 \)

In Problems 43–50, determine the area of the region bounded by the given curve, the x-axis, and the given lines.

43. \( y = x^3, \quad x = 0, \quad x = 2 \)  
44. \( y = 4e^x, \quad x = 0, \quad x = 3 \)  
45. \( y = \sqrt{x} + 4, \quad x = 0 \)  
46. \( y = x^2 - x - 6, \quad x = -4, \quad x = 3 \)  
47. \( y = 5x - x^2 \)  
48. \( y = \sqrt{x}, \quad x = 8, \quad x = 16 \)  
49. \( y = \frac{1}{x} + 2, \quad x = 1, \quad x = 4 \)  
50. \( y = x^3 - 1, \quad x = -1 \)
In Problems 51–58, find the area of the region bounded by the given curves.

51. \( y^2 = 4x, \ x = 0, \ y = 2 \) \hspace{1cm} 52. \( y^3 = 2x^2 - 5, \ x = 0, \ y = 4 \)

53. \( y = x(x - a), \ y = 0 \) for \( 0 < a \)

54. \( y^2 = 2x^2, \ x = y^2 + 9 \) \hspace{1cm} 55. \( y = x^2 - x, \ y = 10 - x^2 \)

56. \( y = \sqrt{x}, \ x = 0, \ y = 3 \) \hspace{1cm} 57. \( y = \ln x, \ x = 0, \ y = 0, \ y = 1 \)

58. \( y = 3 - x, \ y = x - 4, \ y = 0, \ y = 3 \)

59. Marginal Revenue If marginal revenue is given by

\[
\frac{dr}{dq} = 100 - \frac{3}{2} \sqrt{2q}
\]

Determine the corresponding demand equation.

60. Marginal Cost If marginal cost is given by

\[
\frac{dc}{dq} = q^2 + 7q + 6
\]

And fixed costs are 2500, determine the total cost of producing six units. Assume that costs are in dollars.

61. Marginal Revenue A manufacturer’s marginal-revenue function is

\[
\frac{dr}{dq} = 250 - q - 0.2q^2
\]

If \( r \) is in dollars, find the increase in the manufacturer’s total revenue if production is increased from 15 to 25 units.

62. Marginal Cost A manufacturer’s marginal-cost function is

\[
\frac{dc}{dq} = \frac{1000}{\sqrt{3q + 10}}
\]

If \( c \) is in dollars, determine the cost involved to increase production from 10 to 33 units.

63. Hospital Discharges For a group of hospitalized individuals, suppose the discharge rate is given by

\[
f(t) = 0.007e^{-0.007t}
\]

Where \( f(t) \) is the proportion discharged per day at the end of \( t \) days of hospitalization. What proportion of the group is discharged at the end of 100 days?

64. Business Expenses The total expenditures (in dollars) of a business over the next five years are given by

\[
\int_0^5 4000e^{0.05t} dt
\]

Evaluate the expenditures.

65. Find the area of the region between the curves \( y = 9 - 2x \) and \( y = x \) from \( x = 0 \) to \( x = 4 \).

66. Find the area of the region between the curves \( y = 2x^2 \) and \( y = 2 - 5x \) from \( x = -1 \) to \( x = \frac{1}{5} \).

67. Consumers’ and Producers’ Surplus The demand equation for a product is

\[
p = 0.01q^2 - 1.1q + 30
\]

And the supply equation is

\[
p = 0.01q^2 + 8
\]

Determine consumers’ surplus and producers’ surplus when market equilibrium has been established.

68. Consumers’ Surplus The demand equation for a product is

\[
p = (q - 4)^2
\]

And the supply equation is

\[
p = q^2 + q + 7
\]

Where \( p \) (in thousands of dollars) is the price per 100 units when \( q \) hundred units are demanded or supplied. Determine consumers’ surplus under market equilibrium.

69. Biology In a discussion of gene mutation, the equation

\[
\int_{q_0}^{q_1} \frac{dq}{q - a} = -(u + v) \int_0^a dt
\]

Occurs, where \( u \) and \( v \) are gene mutation rates, the \( q \)’s are gene frequencies, and \( n \) is the number of generations. Assume that all letters represent constants, except \( q \) and \( t \). Integrate both sides and then use your result to show that

\[
n = \frac{1}{u + v} \ln \left| \frac{q_0 - q}{q_n - q} \right|
\]

70. Fluid Flow In studying the flow of a fluid in a tube of constant radius \( R \), such as blood flow in portions of the body, we can think of the tube as consisting of concentric tubes of radius \( r \), where \( 0 \leq r \leq R \). The velocity \( v \) of the fluid is a function of \( r \) and is given by

\[
v = \frac{P_1 - P_2)(R^2 - r^2)}{4q l}
\]

Where \( P_1 \) and \( P_2 \) are pressures at the ends of the tube, \( \eta \) (a Greek letter read “eta”) is the fluid viscosity, and \( l \) is the length of the tube. The volume rate of flow through the tube, \( Q \), is given by

\[
Q = \int_0^R 2\pi rv dr
\]

Show that \( Q = \frac{\pi R^4(P_1 - P_2)}{8q l} \). Note that \( R \) occurs as a factor to the fourth power. Thus, doubling the radius of the tube has the effect of increasing the flow by a factor of 16. The formula that you derived for the volume rate of flow is called Poiseuille’s law, after the Frenchphysiologist Jean Poiseuille.

71. Inventory In a discussion of inventory, Barbosa and Friedman refer to the function

\[
g(x) = \frac{1}{k} \int_1^{1/2} kau^k du
\]

Where \( k \) and \( r \) are constants, \( k > 0 \) and \( r > -2 \), and \( x > 0 \). Verify the claim that

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Chapter 13

Integration

1. Find the differential of the following function in terms of $x$ and $dx$:
   \[ y = (9x + 3)e^{2x^3 + 3} \]

2. For the following function, find $\Delta y$ and $dy$ for the given values of $x$ and $dx$:
   \[ y = \ln x; \quad x = 1, \quad dx = 0.01 \]

3. Let $f(x) = x^{3x}$.

4. Demand \quad The demand, $q$, for a monopolist’s product is related to the price per unit, $p$, according to the equation
   \[ 2 + \frac{q^2}{200} = \frac{4000}{p^2} \]
   (a) Verify that 40 units will be demanded when the price per unit is $20.
   (b) Show that $\frac{dq}{dp} = -2.5$ when the price per unit is $20.
   (c) Use differentials and the results of parts (a) and (b) to approximate the number of units that will be demanded if the price per unit is reduced to $19.20$.

5. Find the indefinite integral
   \[ \int \frac{7}{x^4} \, dx \]

6. Find the indefinite integral
   \[ \int \left( 3y^3 - 2y^2 + \frac{e^y}{6} \right) \, dy \]

7. If $y$ satisfies the following conditions, find $y(x)$ for the given value of $x$:
   \[ y' = -x^2 + 2x, \quad y(2) = 1; \quad x = 1 \]

8. Elasticity of Demand \quad The sole producer of a product has determined that the marginal-revenue function is
   \[ \frac{dr}{dq} = 100 - 3q^2 \]
   Determine the point elasticity of demand for the product when $q = 5$. (Hint: First find the demand function.)

9. Find
   \[ \int \frac{4x}{(2x^3 - 7)^{10}} \, dx \]

10. Find
    \[ \int \frac{12x^2 + 4x + 2}{x + x^2 + 2x^3} \, dx \]

11. Life Span \quad If the rate of change of the expected life span $l$ at birth of people born in Egypt can be modeled by
    \[ \frac{dl}{dt} = \frac{12}{2t + 50} \]
    where $t$ is the number of years after 1940 and the expected life span was 63 years in 1940, find the expected life span for people born in 1998.

12. Find
    \[ \int \frac{(\sqrt{x} + 2)^2}{3\sqrt{x}} \, dx \]

13. Find
    \[ \int \sqrt[3]{(8x)^{3/2} + 5} \, dx \]

14. Find
    \[ \int \ln^3 x \, dx \]

15. Revenue Function \quad The marginal-revenue function for a manufacturer’s product is of the form
    \[ \frac{dr}{dq} = \frac{a}{e^q + b} \]
    for constants $a$ and $b$, where $r$ is the total revenue received (in dollars) when $q$ units are produced and sold. Find the demand function, and express it in the form $p = f(q)$. (Hint: Rewrite $dr/dq$ by multiplying both numerator and denominator by $e^{-q}$.)

16. Evaluate the following definite integral by taking the limit of $S_n$. Use the right-hand endpoint of each subinterval. Sketch the graph, over the given interval, of the function to be integrated.
    \[ \int_1^4 (2x + 1) \, dx \]

17. Find
    \[ \int_0^3 f(x) \, dx \]
    without the use of limits, where
    \[ f(x) = \begin{cases} 
    2 & \text{if } 0 \leq x < 1 \\
    4 - 2x & \text{if } 1 \leq x < 2 \\
    5x - 10 & \text{if } 2 \leq x \leq 3 
    \end{cases} \]

18. Evaluate
    \[ \int_9^{36} (\sqrt{x} - 2) \, dx \]

19. Evaluate
    \[ \int_0^1 e^x - e^-x \, dx \]

20. Evaluate
    \[ \int_2^3 \left( \int_2^3 e^x \, dx \right) \, dx \]
    (Hint: It is not necessary to find $\int_2^3 e^x \, dx$.)
21. **Continuous Income Flow**  
The present value (in dollars) of a continuous flow of income of $2000 a year for five years at 6% compounded continuously is given by
\[ \int_{0}^{5} 2000e^{-0.06t} \, dt \]
Evaluate the present value to the nearest dollar.

22. **Hospital Discharges**  
For a group of hospitalized individuals, suppose the discharge rate is given by
\[ f(t) = \frac{81 \times 10^6}{(300 + t)^4} \]
where \( f(t) \) is the proportion of the group discharged per day at the end of \( t \) days. What proportion has been discharged by the end of 700 days?

23. Use a definite integral to find the area of the region bounded by the given curve, the \( x \)-axis, and the given lines. First sketch the region; watch out for areas of regions that are below the \( x \)-axis.

\[ y = 3x^2 - 4x, \quad x = -2, \quad x = -1 \]

24. Express, in terms of a single integral, the total area of the region in the first quadrant bounded by the \( x \)-axis and the graphs of \( y^2 = x \) and \( 2y = 3 - x \). Do not evaluate the integral.

25. The first equation given below is a demand equation and the second is a supply equation of a certain product. In each case, determine consumers’ surplus and producers’ surplus under market equilibrium.

\[ p = \frac{50}{q + 5} \]
\[ p = \frac{q}{10} + 4.5 \]

26. The demand equation for a product is
\[ (p + 10)(q + 20) = 1000 \]
and the supply equation is
\[ q - 4p + 10 = 0 \]

(a) Verify, by substitution, that market equilibrium occurs when \( p = 10 \) and \( q = 30 \).
(b) Determine consumers’ surplus under market equilibrium.

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**EXPLORE & EXTEND**  
**Delivered Price**

Suppose that you are a manufacturer of a product whose sales occur within \( R \) kilometers of your mill. Assume that you charge customers for shipping at the rate \( s \), in dollars per kilometer, for each unit of product sold. If \( m \) is the unit price (in dollars) at the mill, then the delivered unit price \( p \) to a customer \( x \) kilometers from the mill is the mill price plus the shipping charge \( sx \):
\[
p = m + sx \quad 0 \leq x \leq R \tag{1}\]

The problem is to determine the average delivered price of the units sold.

Suppose that there is a function \( f \) such that \( f(t) \geq 0 \) on the interval \([0, R]\) and such that the area under the graph of \( f \) and above the \( t \)-axis from \( t = 0 \) to \( t = x \) represents the total number of units \( Q \) sold to customers within \( x \) km of the mill. [See Figure 13.43(a).] You can refer to \( f \) as the distribution of demand. Because \( Q \) is a function of \( x \) and is represented by area,
\[
Q(x) = \int_{0}^{x} f(t) \, dt
\]

**FIGURE 13.43** Number of units sold as an area.
In particular, the total number of units sold within the market area is

\[
Q(R) = \int_0^R f(t) \, dt
\]

[see Figure 13.43(b)]. For example, if \(f(t) = 10\) and \(R = 100\), then the total number of units sold within the market area is

\[
Q(100) = \int_0^{100} 10 \, dt = 10t\bigg|_0^{100} = 1000 - 0 = 1000
\]

The average delivered price \(A\) is given by

\[
A = \frac{\text{total revenue}}{\text{total number of units sold}}
\]

Because the denominator is \(Q(R)\), \(A\) can be determined once the total revenue is found.

To find the total revenue, first consider the number of units sold over an interval. If \(t_1 < t_2\) [see Figure 13.44(a)], then the area under the graph of \(f\) and above the \(t\)-axis from \(t = 0\) to \(t = t_1\) represents the number of units sold within \(t_1\) kilometers of the mill. Similarly, the area under the graph of \(f\) and above the \(t\)-axis from \(t = 0\) to \(t = t_2\) represents the number of units sold within \(t_2\) kilometers of the mill. Thus

\[
Q(t_2) - Q(t_1) = \int_{t_1}^{t_2} f(t) \, dt
\]

For example, if \(f(t) = 10\), then the number of units sold to customers located between 4 and 6 kilometers of the mill is

\[
Q(6) - Q(4) = \int_4^6 10 \, dt = 10t\bigg|_4^6 = 60 - 40 = 20
\]

The area of the shaded region in Figure 13.44(a) can be approximated by the area of a rectangle [see Figure 13.44(b)] whose height is \(f(t)\) and whose width is \(dt\), where \(dt = t_2 - t_1\). Thus the number of units sold over the interval of length \(dt\) is approximately \(f(t)dt\). Because the price of each of these units is [from Equation (1)] approximately \(m + st\), the revenue received is approximately

\[
(m + st)f(t)dt
\]

The sum of all such products from \(t = 0\) to \(t = R\) approximates the total revenue. Definite integration gives

\[
\sum (m + st)f(t)dt \rightarrow \int_0^R (m + st)f(t)dt
\]

Thus,

\[
\text{total revenue} = \int_0^R (m + st)f(t)dt
\]

Consequently, the average delivered price \(A\) is given by

\[
A = \frac{\int_0^R (m + st)f(t)dt}{Q(R)}
\]

Equivalently,

\[
A = \frac{\int_0^R (m + st)f(t)dt}{\int_0^R f(t)dt}
\]

For example, if \(f(t) = 10\), \(m = 200\), \(s = 0.25\), and \(R = 100\), then

\[
\int_0^R (m + st)f(t)dt = \int_0^{100} (200 + 0.25t) \cdot 10 \, dt
\]

\[
= 10 \int_0^{100} (200 + 0.25t) \, dt
\]

\[
= 10 \left( 200t + \frac{t^2}{8} \right)\bigg|_0^{100}
\]

\[
= 10 \left[ 20,000 + \frac{10,000}{8} \right] - 0
\]

\[
= 212,500
\]

From before,

\[
\int_0^R f(t) dt = \int_0^{100} 10 \, dt = 1000
\]

Thus, the average delivered price is

\[
212,500/1000 = $212.50.
\]
**Problems**

1. If \( f(t) = 100 - 2t \), determine the number of units sold to customers located (a) within 5 km of the mill, and (b) between 20 and 25 km of the mill.

2. If \( f(t) = 40 - 0.5t \), \( m = 50 \), \( s = 0.20 \), and \( R = 80 \), determine (a) the total revenue, (b) the total number of units sold, and (c) the average delivered price.

3. If \( f(t) = 900 - t^2 \), \( m = 100 \), \( s = 1 \), and \( R = 30 \), determine (a) the total revenue, (b) the total number of units sold, and (c) the average delivered price. Use a graphing calculator if you like.

4. How do real-world sellers of such things as books and clothing generally determine shipping charges for an order? (Visit an online retailer to find out.) How would you calculate average delivered price for their products? Is the procedure fundamentally different from the one discussed in this Explore & Extend?
Absolute extrema | القيم المقصودة
We refer to either a absolute maximum or a absolute minimum as a absolute extremum (plural: absolute extrema).

Absolute maximum | الحد الأقصى
A function $f$ has an absolute maximum at $a$ if $f(a) \geq f(x)$ for all $x$ in the domain of $f$.

Absolute minimum | ححد الأدنى
$f$ has an absolute minimum at $a$ if $f(a) \leq f(x)$ for all $x$ in the domain of $f$.

Absolute Value | قيمة مطلقة
On the real numbers line the absolute value is the distance of a number $x$ from 0.

law of mutually exclusive events | قانون الأحداث المتميزة
If $E$ and $F$ are mutually exclusive events then $P(E \cup F) = P(E) + P(F)$.

Amortization schedules | جدول السداد
A table giving an analysis of how each payment in the loan is handled.

Amortizing | إتقان أقساط/إطلاق القرض
A loan, such as a mortgage, is amortized when part of each installment payment is used to pay interest and the remaining part is used to reduce the principal.

Annuity | فائدة مسننة
A finite sequence of payments made at fixed periods of time over a given interval.

Annuity due | الأقساط المستدفة
is an annuity in which each payment is made at the beginning of a payment period.

Antiderivative | اشتقاق عكسي
An antiderivative of $f$ is a function whose derivative is $f$.

Area between the curves | المنطقة المحصورة بين المنحنين
An arithmetic sequence is a sequence $(bk)$ defined recursively by $b_1 = a$ and, for each positive integer $k$, $bk+1 = d + bk$ for fixed real numbers $a$ and $d$.

Artificial objective function | دالة الهيكل الأسطوانة
A function of the form $W = Z - Mt$ where $z$ is the original objective function, $t$ is the artificial variable and the constant $M$ is a very large positive number.

Artificial Variables | المتغيرات الإضافية
A nonnegative variable added to the left side of the equation in which the coefficient of the slack variable is $-1$.

Asymptote | الخط الممتد
An asymptote is a line that a curve approaches arbitrarily close without crossing it.

Augmented matrix | المصفوفة الممتدة
Obtained by adding an additional column for the constants to the right of the coefficient matrix.

Average total cost | متوسط التكلفة الكلية
Total cost divided by the quantity produced.

Average rate of change | متوسط معدل التغير
The slope of the secant line connecting two given points.

Average value of a function | قيمة الأداء
over an interval $[a, b]$ is denoted by $f$ and is given by

\[ \bar{f} = \frac{1}{b-a} \int_{a}^{b} f(x)dx \]

Axis of symmetry | محور التماثل
parabola is symmetric about a vertical line, called the axis of symmetry of the parabola.

Basic Counting Principle | صيغة بالايز
Suppose that a procedure involves a sequence of $k$ stages. Let $n_1$ be the number of ways the first can occur and $n_2$ be the number of ways the second can occur. Continuing in this way, let $n_k$ be the number of ways the $k$th stage can occur. Then the total number of different ways the procedure can occur is $n_1 \cdot n_2 \cdot \cdots \cdot n_k$.

Bayes’s formula | صيغة بيرس
If $E$ is an event and $F_1, F_2, \ldots, F_n$ is a partition then, to find the conditional probability of event $F_i$, given $E$, when prior and conditional probabilities are known, we can use Bayes’s formula.

Bayes’s probability tree | شجرة الاحتمال
Used to solve a Bayes-type problem.

Bernoulli trials | تجربة برنولي
Whenever we have $n$ independent trials of an experiment in which each trial has only two possible outcomes (success and failure) and the probability of success in each trial remains the same, the trials are called Bernoulli trials.

Binomial Distribution | توزيع احتمالي ثنائي
In a Binomial experiment, if $X$ is the number of successes in $n$ trials, then the distribution of $X$ is called a binomial distribution.

Binomial experiment | تجربة ذات الحدين
If there are only two possible outcomes (success and failure) for each independent trial, and the probabilities of success and failure do not change from trial to trial, then the experiment is called a binomial experiment.
Bounded feasible region  |  منطقة عملية محدودة
A feasible region that can be contained within a circle

Break – even point  |  نقطة تعادل
The point at which total revenue equals total cost

Certain event  |  الحد الأدنى
The sample space itself. It will then occur with a probability 1

Change-of-base formula  |  تغيير الصيغة الأساسية
\( \log_b m = \frac{(\log_a m)}{(\log_a b)} \)

Closed interval  |  الفترة المغلقة
The set of all real numbers \( x \) for which \( a \leq x \leq b \) and includes the numbers \( a \) and \( b \),

Coefficient matrix  |  مصفوفة المعامدات
Matrix formed by the coefficients of the variables in each equation. The first column consists of the coefficients of the first variable, the second one to the second variable and so on

Combination  |  تجميع
A selection of \( r \) objects, without regard to order and without repetition, selected from \( n \) distinct objects

Common Logarithm  |  اللوغاريتم المشترك
Logarithm to the base 10.

Competitive products  |  تنافس المنتجات
A and \( B \) are competitive products or substitutes if an increase in the price of \( B \) causes an increase in the demand for \( A \), if it is assumed that the price of \( A \) does not change. Similarly, an increase in the price of \( A \) causes an increase in the demand for \( B \) when the price of \( B \) is held fixed.

Complement  |  متمم
The event consisting of all sample points in the sample space that are not in the event

Complementary products  |  سلسمثلكة
A and \( B \) are Complementary products if an increase in the price of \( B \) causes a decrease in the demand for \( A \) if the price of \( A \) does not change. Similarly, an increase in the price of \( A \) causes a decrease in the demand for \( B \) when the price of \( B \) is held fixed.

Composite function  |  الدالة المركبة
Given two functions \( f \) and \( g \), the operation of applying \( g \) and then applying \( f \) to the result is called composition and the resulting function, denoted \( f \circ g \), is called the composite of \( f \) with \( g \).

Compound amount  |  المبلغ المركّم
The original principal invested or borrowed plus all accrued interest

Compound interest  |  فائدة مركبة
The difference between the compound amount and the original principal

Compounded continuously  |  مركّب بشكل مستمر
If the number of interest periods per year, is increased indefinitely then the length of each period approaches 0 and we say that interest is compounded continuously.

Concave down  |  مقعرة لأعلى
\( f \) is said to be concave up down on \( (a, b) \) if \( f' \) is decreasing on \( (a, b) \).

Concave up  |  مقعرة لأعلى
\( f \) is said to be concave up on \( (a, b) \) if \( f' \) is increasing on \( (a, b) \).

Conditional probability  |  الاحتمال المشروط
The probability of an event when it is known that some other event has occurred.

Constant function  |  دالة ثابتة
A function of the form \( h(x) = c \), where \( c \) is a constant

Constraint  |  تحديدات، حصر
Restrictions on the domain

Consumers’ surplus  |  فائض المستهلك
The total gain to consumers who are willing to pay more than the equilibrium price

Consumption function  |  دالة الاستهلاك
The function relating consumption to income

Continuous at a point  |  مستمرة
A function \( f \) is continuous at \( a \) if and only if the following three conditions are met:
1. \( f(a) \) exists
2. \( \lim_{x \to a} f(x) \) exists
3. \( \lim_{x \to a} f(x) = f(a) \)

Continuous function  |  دالة مستمرة
Continuous at each point of the domain

Continuous random variable  |  متغير عشوائي مستمر
A random variable a continuous random variable if it assumes all values in some interval or intervals

Convergent  |  متقارب
The improper integral is convergent if its value is a finite number

Corner point  |  نقطة الزاوية
A linear function defined on a nonempty, bounded feasible region has a maximum (minimum) value, and this value can be found at a corner point.

Critical point  |  النقطة الحادة
If \( a \) is a critical value for a function \( f \) then \( (a, f(a)) \) is a critical point

Critical value  |  قيمة الحافة
For \( a \) in the domain of \( f \), if either \( f(a) = 0 \) or \( f(a) \) does not exist, then \( a \) is called a critical value for \( f \).

Current assets  |  الأصول الجاريّة
Are assets such as cash, merchandise inventory, and accounts receivable to its current liabilities
The Arab World Edition of Haeussler’s widely used Introductory Mathematical Analysis for Business, Economics, and Life and Social Sciences combines solid pedagogical features with simple, direct language to give readers an excellent introduction to tertiary mathematics. The use of local images and case studies taken from well-known regional companies—such as Aramex, Etisalat, and Emaar—helps Arab readers apply mathematical concepts to their environment. Step-by-step answers throughout, different levels of exercises, and biographies of famous mathematicians make this a highly practical and user-friendly textbook for undergraduate students of mathematics, business, and social sciences in the Arab region and beyond.

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